Remarks on zeta regularized products

Kazufumi KIMOTO and Masato WAKAYAMA

Abstract

In constructing functions having a certain invariance and a given set of zeros, the zeta regularized product $\prod$ and its natural generalization $\dot{\prod}$ play important roles. To deal with wider class of sequences, we introduce an extended version $\ddot{\prod}_n a_n$ of such regularizations. This allows us to treat the case where the attached zeta function of $\{a_n\}_n$ has even a log-singularity at the origin. We discuss several examples of the type $\ddot{\prod}_n \varphi(a_n - x)$ for choosing $a_n = n$, $a_n =$ the essential zeros ρ of zeta functions, etc., especially for the trigonometric functions φ. As one of the applications, we give a criterion for the validity of a distribution formula for the essential zeros of $\zeta(s)$ in terms of $\ddot{\prod}$, which is a weaker version of the Riemann Hypothesis (RH).
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1 Introduction

For a given sequence $\boldsymbol{a} = \{a_n\}_{n\in I}$ of non-zero complex numbers, the **zeta regularized product** $\prod_{n\in I} a_n$ of $\boldsymbol{a}$ is defined by

$$\prod_{n\in I} a_n := \exp\left(-\partial_s \zeta_{\boldsymbol{a}}(s)\big|_{s=0}\right) \tag{1.1}$$

when the attached zeta function $\zeta_{\boldsymbol{a}}(s) := \sum_{n\in I} a_n^{-s}$ is analytically continued to some region containing the origin $s=0$ and holomorphic at $s=0$ (see, e.g. [D2]). Here ∂_s denotes the partial differential operator with respect to s.

In constructing functions having a certain invariance and a given set of zeros, the zeta regularized product plays important roles. Particularly, the zeta regularized product $\prod$ defines a determinant of an operator A by $\det A := \prod_n \lambda_n$ where λ_n denotes the eigenvalue of A. For instance, a Selberg zeta function $Z_\Gamma(s)$ (see Section 5.3) has a determinant expression via the Laplacian Δ_Γ of the Riemann surface (see, e.g. [V]). Hence the analogue of the Riemann Hypothesis of $Z_\Gamma(s)$ follows from the determinant expression because Δ_Γ is positive definite. All zeta functions which satisfy an analogue of the Riemann Hypothesis are known to be having such determinant expressions. The most important question is whether one can associate a determinant expression to a given zeta function via some self (skew-)adjoint operator. As to the Riemann zeta function, there is a deep observation [D1], [D2] (see also [KuOW] for some trial) in this direction.

Among various features of zeta regularized products, we focus our attention on the functional aspect in this paper; zeta regularization methods often allow us to express a function in very transparent (or rather intuitive) manner as well as to construct a function equipped with a certain invariance such as (quasi-)periodicity. For instance, the function

$$S_{\mathbb{Z}}(x) := \prod_{n\in\mathbb{Z}} (n-x)$$

essentially gives the sine function and hence has a periodicity as is expected from its form (see Example 3.2). It is also seen that the zeros of $S_{\mathbb{Z}}(x)$ are exactly given by $x=n$ $(n\in\mathbb{Z})$. In general, for a given sequence $\boldsymbol{a}$ and a good function φ, we may expect that the product

$$D_{\boldsymbol{a}}(x;\varphi) := \prod_{n\in I} \varphi(a_n - x)$$

(if it exists) defines a function whose zeros are exactly given by the set

$$\coprod_{n\in I} \{x\in\mathbb{C} \mid \varphi(a_n - x) = 0\}$$

and is piecewise holomorphic.

However, the situation we can apply the zeta regularization method is rather restricted. For instance, if we take a geometric sequence $\boldsymbol{a} = \{q^n\}_{n\geq 0}$ $(q > 1)$ typically, then the attached Dirichlet series

$$\zeta_{\boldsymbol{a}}(s) := \sum_{n\geq 0} a_n^{-s} = \frac{1}{1-q^{-s}}$$

is analytically continued to the whole s-plane but it has a simple pole at the origin $s = 0$. This shows that the zeta regularized product $\prod_n q^n$ is not defined. In order to handle such cases, an extended notion called a **dotted product** has been introduced in [KuW2] (see also [I]). This dotted product $\boxdot$ is actually defined by

$$\text{(1.2)} \qquad \underset{n\in I}{\boxdot}\, a_n := \exp\left(-\operatorname*{Res}_{s=0} \frac{\zeta_{\boldsymbol{a}}(s)}{s^2}\right)$$

for a sequence $\boldsymbol{a} = \{a_n\}_{n\in I}$ when the (analytically continued) zeta function $\zeta_{\boldsymbol{a}}(s)$ is meromorphic at the origin $s = 0$. Notice that this dotted product provides a generalization of the original regularized product since $\zeta_{\boldsymbol{a}}'(0) = \mathrm{Res}_{s=0}\, \zeta_{\boldsymbol{a}}(s)/s^2$ if $\zeta_{\boldsymbol{a}}(s)$ is holomorphic at $s = 0$. We also remark that this definition of a dotted product is still applicable when the origin $s = 0$ is an isolated singularity of $\zeta_{\boldsymbol{a}}(s)$ (see Remark 3.1). By using this new regularization, we can treat, for instance, regularized products of the values of trigonometric functions and those of q-numbers over the lattice $\mathbb{Z}$ and the semi-lattice $\mathbb{Z}_{\geq 0}$, which allow us to construct easily a function having some translation property such as periodicity. These products are mainly exhibited in Sections 5.1 (see also Remark 5.4).

Still, there exist natural situations we need a further extension of the zeta regularization $\boxdot$. Let us show such an example. It is well-known that if we put $\hat{\zeta}(s) := \zeta(s)\pi^{s/2}\Gamma(s/2)$ then the functional equation of the Riemann zeta function can be written in a symmetric way; $\hat{\zeta}(1-s) = \hat{\zeta}(s)$. Let $\zeta_{l\infty}(s)$ be the higher Riemann zeta function defined by $\zeta_{l\infty}(s) := \prod_{n\geq 1} \zeta(s+ln)$ [KuMW]. Then, in the course of the study for obtaining a symmetric functional equation of $\zeta_{l\infty}(s)$ similarly, it is quite helpful to introduce a function defined (naively) by

$$\text{(1.3)} \qquad S_l(x) := \underset{\mathrm{Im}(\rho)>0}{\boxast} \sin\frac{\pi(\rho - x)}{l},$$

especially, in order to determine an explicit form of the completion $\hat{\zeta}_{l\infty}(s)$ of $\zeta_{l\infty}(s)$ (see Example 5.1). Here $\boxast$ denotes a suitably formulated product over the non-trivial zeros ρ of the Riemann zeta function $\zeta(s)$ in the upper half plane. One possibility for defining such a function is to employ a zeta regularized product as the product $\boxed{\ast}$ in (1.3). However, in the case of $S_l(x)$, the associated zeta function

$$L_l(s,x) := \sum_{\mathrm{Im}(\rho)>0} \left\{ \sin \frac{\pi(\rho - x)}{l} \right\}^{-s} \tag{1.4}$$

has a log-singularity at $s = 0$ according to the famous result of Cramér [Cr] in 1919. In order to overcome such difficulties, we introduce much further generalization $\boxed{:}$ of zeta regularized products described above (see Section 2). We show that a function defined via this new zeta regularized product has a Weierstrass canonical product expression, that is, it has a desired set of zeros counting with multiplicity like in [V], [I], [KiKuSW1].

In Section 4, we deal with various regularized products of the values of trigonometric functions. In this trigonometric case, the presence of a differential equation is useful for the discussion. In particular, as applications of the regularized product $\boxed{:}$, we provide several interesting examples in Section 5 relating such as the Riemann Hypothesis (Theorem 5.4), the Selberg's 1/4-conjecture (Theorem 5.5 and Remark 5.7), the determinant of the trigonometric function of eigenvalues of a Laplacian of a Riemann surface and certain q-analogues connected with the Jackson q-gamma function (see, e.g. [AAR]), etc.

Furthermore, we make an experimental study concerning the 'regularized product' of the values of the elliptic theta functions in Section 6. We propose a candidate of a suitable regularization and show that the 'regularized product' of the theta functions $\vartheta(x+nt,t)$ over the lattice $\mathbb{Z}$ produces essentially the theta function again while the direction of periodicity and that of quasi-periodicity are switched. This result is immediately extended to the so-called Jacobi forms (see, e.g. [EZ]).

We hope also that in general one may use a difference-differential equation of φ (if any) to discuss a product of $\varphi(a_n - x)$'s. A part of Section 6 is devoted to give a small calculation about $\wp(z)$ as an example of such a situation. In the last position we remark on the construction of certain new zeta extensions in the sense of [KuW1] by means of regularized products.

Convention

In this paper we distinguish three kinds of product symbols $\prod$, $\boxed{\cdot}$ and $\boxed{:}$ in order to specify which regularization we actually need for a given sequence. We also use the symbol $\boxed{\ast}$ to indicate a regularized product which is neither specified nor formulated suitably.

Throughout the paper we fix the log-branch by

$$\log z = \log |z| + i \arg z \quad (-\pi \leq \arg z < \pi)\,. \tag{1.5}$$

Remark that the values of zeta regularized products depend on the choice of the log-branch.

We denote by $\mathbb{C}$ the entire complex plane, $\mathbb{R}$ the real axis, $\mathbb{Z}$ the lattice consisting of all rational integers and $\mathbb{Z}_{\geq 0}$ the semi-lattice consisting of non-negative integers. We also denote by $\mathbb{Q}$ the rational number field.

2 Double-dotted products

In this section we introduce the notion of a **double-dotted product** or a **ddotted product** in short, which is a generalization of a zeta regularized product [•]. Employing this regularized product, we can treat the case where the attached Dirichlet series has even a log-singularity at the origin.

2.1 Definition of ddotted products

Let $\boldsymbol{a} = \{a_n\}_{n\in I}$ be a sequence of non-zero complex numbers. We define the associated zeta function (or the Dirichlet series) $\zeta_{\boldsymbol{a}}(s)$ of $\boldsymbol{a}$ by

$$\zeta_{\boldsymbol{a}}(s) := \sum_{n\in I} a_n^{-s}$$

which is supposed to be convergent absolutely for $\mathrm{Re}(s) \gg 1$.

Assume that there exists a finite collection of functions $\{Q_m(s;\boldsymbol{a})\}_{m=1}^{M}$ which are meromorphic around the origin $s = 0$ such that the difference

$$P(s;\boldsymbol{a}) := \zeta_{\boldsymbol{a}}(s) - \sum_{m=1}^{M} Q_m(s;\boldsymbol{a})(\log s)^m$$

is analytically continued to the some region containing the origin as a single-valued meromorphic function. We also suppose that the zeta function $\zeta_{\boldsymbol{a}}(s)$ itself is also analytically continued to the right half plane $\mathrm{Re}(s) > 0$. Then we say $\zeta_{\boldsymbol{a}}(s)$ (and also the sequence $\boldsymbol{a}$) is **regularizable**, and we call $P(s;\boldsymbol{a})$ the **meromorphic part** of $\zeta_{\boldsymbol{a}}(s)$ at $s = 0$. The order of the (possible) pole of $P(s;\boldsymbol{a})$ at the origin $s = 0$ is called a **depth** of $\zeta_{\boldsymbol{a}}(s)$.

When the zeta function $\zeta_{\boldsymbol{a}}(s)$ is regularizable, we define its **linear term** at $s = 0$ by

$$\underset{s=0}{\mathcal{LT}}\, \zeta_{\boldsymbol{a}}(s) := \underset{s=0}{\mathrm{Res}}\, \frac{P(s;\boldsymbol{a})}{s^2}.$$

In terms of this linear term $\mathcal{LT}_{s=0}\,\zeta_{\boldsymbol{a}}(s)$ of a given zeta function $\zeta_{\boldsymbol{a}}(s)$, we define the following extended version of zeta regularized products.

Definition 2.1 (Ddotted regularization). Let $\boldsymbol{a} = \{a_n\}_{n\in I}$ be a regularizable sequence. Define the **ddotted product** of $\boldsymbol{a}$ by

$$\prod\!\!\!\!\!\!\!\;\colon_{n\in I} a_n := \exp\left(-\mathop{\mathcal{LT}}_{s=0}\zeta_{\boldsymbol{a}}(s)\right). \tag{2.1}$$

Here $\zeta_{\boldsymbol{a}}(s)$ is the associated zeta function of $\boldsymbol{a}$.

It is easy to see that the meromorphic part of a given regularizable zeta function is uniquely determined once we fix the branch of s. Namely, this truncation procedure is legitimate. Hence the definition of the ddotted regularized product is well-defined.

Remark 2.1. It is readily observed that the ddotted product $\prod\!\!\!\!\!\!\;\colon_n a_n$ of $\boldsymbol{a} = \{a_n\}_n$ is nothing but the dotted product $\prod\!\!\!\!\!\!\;\cdot_n a_n$ of $\boldsymbol{a}$ when the attached zeta function $\zeta_{\boldsymbol{a}}(s)$ is meromorphic at $s = 0$ (i.e. $\zeta_{\boldsymbol{a}}(s) = P(s;\boldsymbol{a})$).

The following proposition is elementary but quite important.

Proposition 2.1. *When all of the appearing ddotted products exist, we have*

$$\prod\!\!\!\!\!\!\;\colon_{n\in I\sqcup J} a_n = \prod\!\!\!\!\!\!\;\colon_{n\in I} a_n \prod\!\!\!\!\!\!\;\colon_{n\in J} a_n, \tag{2.2}$$

$$\prod\!\!\!\!\!\!\;\colon_{n\in I} a_n^k = \left(\prod\!\!\!\!\!\!\;\colon_{n\in I} a_n\right)^k, \tag{2.3}$$

$$\prod\!\!\!\!\!\!\;\colon_{n\in I} \lambda a_n = \exp\left(-\sum_{n=1}^{\mu+1}\frac{(-\log\lambda)^n}{n!}\mathop{\mathcal{LT}}_{s=0} s^n\zeta_{\boldsymbol{a}}(s)\right)\prod\!\!\!\!\!\!\;\colon_{n\in I} a_n, \tag{2.4}$$

$$\prod\!\!\!\!\!\!\;\colon_{n\in I} \overline{a_n} = \overline{\prod\!\!\!\!\!\!\;\colon_{n\in I} a_n}, \tag{2.5}$$

for $k > 0$ and $\lambda > 0$. Here μ in (2.4) *denotes the depth of the attached zeta function $\zeta_{\boldsymbol{a}}(s)$ of the sequence $\boldsymbol{a} = \{a_n\}_{n\in I}$ and $\overline{z}$ the complex conjugate of z.*

Proof. The formulas (2.2) and (2.3) are immediate by the definition of the ddotted products. The formula (2.4) is obtained by a similar discussion in [KiKuSW1]. The formula (2.5) follows from the equality $\zeta_{\bar{\boldsymbol{a}}}(s) = \overline{\zeta_{\boldsymbol{a}}(\overline{s})}$ where $\bar{\boldsymbol{a}} = \{\overline{a_n}\}_{n\in I}$ is the complex conjugate of $\boldsymbol{a} = \{a_n\}_{n\in I}$. □

Remark 2.2. In general, two regularized products $\prod\!\!\!\!\!\!\;\colon_{n\in I} a_nb_n$ and $\prod\!\!\!\!\!\!\;\colon_{n\in I} a_n \prod\!\!\!\!\!\!\;\colon_{n\in I} b_n$ are different. Actually, there exists an anomaly between them.

2.2 Functions defined by zeta regularizations

Let φ be a meromorphic function and $\boldsymbol{a} = \{a_n\}_{n\in I}$ be a sequence of complex numbers. We are interested in the function of the form

$$D_{\boldsymbol{a}}(x;\varphi) := \mathop{\prod\!\!\!\!\!\!\!\!\;\;:}_{n\in I} \varphi(a_n - x). \tag{2.6}$$

Using this zeta regularization, we can treat much wider class of functions φ. We assume that the sequence $\boldsymbol{a}$ satisfies $\varphi(a_n) \neq 0$ for any $n \in I$; we call this assumption the **zero-free condition**. This assumption is not essential but for simplicity. Actually, from the formula (2.2), it is clear that one can easily remove a finite number of exceptions of a_n's in the regularized product of $\boldsymbol{a}$. Namely, if we take a finite subset $E \subset I$, then the ddotted product $\mathop{\prod\!\!\!\!\!\!\!\!\;\;:}_{n\in I}(a_n - x)$ is divided as

$$\mathop{\prod\!\!\!\!\!\!\!\!\;\;:}_{n\in I} \varphi(a_n - x) = \prod_{n\in E} \varphi(x - a_n) \times \mathop{\prod\!\!\!\!\!\!\!\!\;\;:}_{n\in I\setminus E} \varphi(x - a_n).$$

We denote by $\zeta(s, x; \boldsymbol{a}; \varphi)$ the associated zeta function

$$\zeta(s, x; \boldsymbol{a}; \varphi) = \sum_{n\in I} \varphi(a_n - x)^{-s},$$

and assume that $\zeta(s, x; \boldsymbol{a}; \varphi)$ is regularizable for a generic $x \in \mathbb{C}$. Precisely, if $\mathrm{Re}(s) \gg 1$, then the Dirichlet series $\zeta(s, x; \boldsymbol{a}; \varphi)$ converges absolutely and uniformly (as a function with respect to x) for each compact subset of $\mathbb{C}$ which does not contain any zeros of $\varphi(a_n - x)$. We also denote by $P(s, x; \boldsymbol{a}; \varphi)$ the meromorphic part of $\zeta(s, x; \boldsymbol{a}; \varphi)$.

Remark 2.3. By definition it is easy to see that the operations $\mathcal{LT}$ and ∂_x is compatible, that is, we have

$$\mathop{\mathcal{LT}}_{s=0}\Big(\partial_x \zeta(s, x; \boldsymbol{a}; \varphi)\Big) = \partial_x\Big(\mathop{\mathcal{LT}}_{s=0} \zeta(s, x; \boldsymbol{a}; \varphi)\Big)$$

for a regularizable zeta function $\zeta(s, x; \boldsymbol{a}; \varphi)$.

In general, we cannot say anything about properties of the function $D_{\boldsymbol{a}}(x;\varphi)$ defined by a zeta regularized product (2.6) a priori. However, we frequently observe that the function $D_{\boldsymbol{a}}(x;\varphi)$ is piecewise holomorphic. More precisely, we have

A typical situation. There exist several connected domains $\{U_j\}_j$ such that $D_{\boldsymbol{a}}(x;\varphi) := \mathop{\prod\!\!\!\!\!\!\!\!\;\;:}_{n\in I} \varphi(a_n - x)$ gives a holomorphic function on each domain U_j but is discontinuous on the boundary ∂U_j of U_j. This discontinuity is originated from the fact that the logarithmic function $\log x$ is multi-valued. If we denote by $D_{\boldsymbol{a}}(x;\varphi;U_j) := \mathop{\prod\!\!\!\!\!\!\!\!\;\;:}_{n\in I} \varphi(a_n - x)\big|_{U_j}$ the restriction of

$D_{\boldsymbol{a}}(x;\varphi)$ on U_j, then each $D_{\boldsymbol{a}}(x;\varphi;U_j)$ is analytically continued to the whole x-plane as an entire function. We denote by $\tilde{D}_{\boldsymbol{a}}(x;\varphi;U_j) = :\mathop{\prod\!\!\!\!\!\!\vdots}_{n\in I}\varphi(a_n - x):_{U_j}$ the extension of $D_{\boldsymbol{a}}(x;\varphi;U_j)$ in order to distinguish this function from the original function $D_{\boldsymbol{a}}(x;\varphi) = \mathop{\prod\!\!\!\!\!\!\vdots}_{n\in I}\varphi(a_n - x)$, and call it the **normal product** associated with the **initial domain** U_j. Remark that two functions $\tilde{D}_{\boldsymbol{a}}(x;\varphi;U_i)$ and $\tilde{D}_{\boldsymbol{a}}(x;\varphi;U_j)$ are not the same function in general if $U_i \cap U_j = \emptyset$, but their difference may be an elementary factor such as the exponential of a polynomial function. For a typical example, see Example 3.2. See also Figure 1 (which shows the case where φ is periodic — the most interesting case). We suppress the symbol U_j and simply write $\tilde{D}_{\boldsymbol{a}}(x;\varphi) = :\mathop{\prod\!\!\!\!\!\!\vdots}_{n\in I}\varphi(a_n - x):$ when the initial domain U_j is clear from the context.

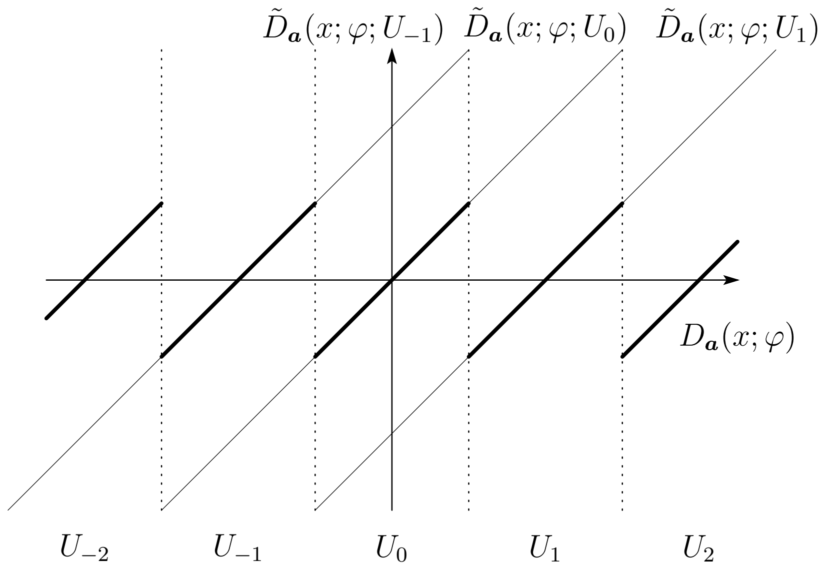

Figure 1: A typical situation (conceptual)

The graph of $D_{\boldsymbol{a}}(x;\varphi)$ is drawn by thick lines. Each restriction $D_{\boldsymbol{a}}(x;\varphi;U_j)$ of $D_{\boldsymbol{a}}(x;\varphi)$ on U_j is holomorphic on U_j and has an extension $\tilde{D}_{\boldsymbol{a}}(x;\varphi;U_j)$ which is entire.

For instance, when $\varphi(z) = z$ or $\sinh z$, the function $D_{\boldsymbol{a}}(x;\varphi)$, if it exists, is actually continued to the whole plane as an entire function. See Sections 3 and 4 for details.

Remark 2.4. When $\zeta(s,x;\boldsymbol{a};\varphi)$ is meromorphic (i.e. $\zeta(s,x;\boldsymbol{a};\varphi) = P(s,x;\boldsymbol{a};\varphi)$), the ddotted regularized product coincides with the dotted product; recall the definition (1.2) of a dotted product.

3 Linear products

Before going to discuss interesting cases of dotted and ddotted products, for the sake of understanding the situation well, we give here some examples when φ is a linear function.

Example 3.1. By the Lerch's calculation [L] in 1894 concerning the Hurwitz zeta function $\zeta(s,x) := \sum_{n=0}^{\infty}(n+x)^{-s}$, we have the zeta regularized product expression of the gamma function

$$\frac{\sqrt{2\pi}}{\Gamma(x)} = \prod_{n=0}^{\infty}(n+x).$$

This formula holds for $x \in \mathbb{C} \setminus \mathbb{Z}$ and integral points $x = -n$ $(n = 0, 1, 2, \dots)$ are removable singularity; in fact, we see that $\lim_{x\to -k} \prod_{n=0}^{\infty}(n+x) = 0$ for $k = 0, 1, 2, \dots$. In this case the regularized product $\prod_{n\geq 0}(n+x)$ itself defines an entire function. We notice that this expression respects the location of zeros in a very apparent way and the functional equation is intuitively understood:

$$\frac{\sqrt{2\pi}}{\Gamma(x)} = \prod_{n=0}^{\infty}(n+x) = x\prod_{n=1}^{\infty}(n+x) = x\frac{\sqrt{2\pi}}{\Gamma(x+1)} \implies \Gamma(x+1) = x\Gamma(x).$$

Some other properties of $\Gamma(x)$ derived easily from the expression such as the multiplication formula of Gauss-Legendre, see [KiKuSW2]. $\square$

As we mentioned in Section 2.2, the function $D_{\boldsymbol{a}}(x) = \prod_{n\in I}(a_n - x)$ does not define an entire function in general but a piecewise holomorphic function. However, if we restrict the function $D_{\boldsymbol{a}}(x)$ on a certain domain $U \subset \mathbb{C}$ and denote it by $D_{\boldsymbol{a}}(x;U)$ so that $D_{\boldsymbol{a}}(x;U)$ is holomorphic on U, then $D_{\boldsymbol{a}}(x;U)$ is continued to the whole x-plane and the extension $\tilde{D}_{\boldsymbol{a}}(x;U)$ is an entire function (see Theorem 3.1 below). A typical example is as follows (cf. Figure 1).

Example 3.2. The ring sine function $S_{\mathbb{Z}}(x)$ of $\mathbb{Z}$

$$S_{\mathbb{Z}}(x) := \prod_{n\in\mathbb{Z}}(n-x) = \begin{cases} 1 - e^{2\pi i x} & x \in U^+ \\ 1 - e^{2\pi i x} & x \in \mathbb{R}\setminus\mathbb{Z} \\ 1 - e^{-2\pi i x} & x \in U^- \end{cases}$$

essentially gives the sine function and hence (piecewisely) has a periodicity as is expected from its form (see [KuMOW]). Here we put $U^+ = \{x \in \mathbb{C} \mid \operatorname{Im} x > 0\}$ and $U^- = \{x \in \mathbb{C} \mid \operatorname{Im} x < 0\}$. The function $S_{\mathbb{Z}}(x)$ is holomorphic in U^+ and U^- respectively, and discontinuous on $\mathbb{R}\setminus\mathbb{Z}$ (piecewise holomorphic function on $\mathbb{C}$). The integral points $x = k$ $(k \in \mathbb{Z})$

are removable singularities; in fact, we see that $\lim_{x\to k} S_{\mathbb{Z}}(x) = 0$. The canonical product expression of $S_{\mathbb{Z}}(x)$ is expressed as

$$S_{\mathbb{Z}}(x) = x\exp\big(f(x;U^{\pm})\big)\prod_{n\neq 0}\Big(1-\frac{x}{n}\Big)\exp\Big(\frac{x}{n}\Big) \qquad (x\in U^{\pm})$$

where the polynomial function $f(x;U^{\pm})$ is given by

$$f(x;U^{+}) = -\pi i x + \log 2\pi + \frac{1}{2}\pi i,$$
$$f(x;U^{-}) = -3\pi i x + \log 2\pi - \frac{1}{2}\pi i.$$

This shows that two entire functions $\tilde{S}_{\mathbb{Z}}(x;U^{+})$ and $\tilde{S}_{\mathbb{Z}}(x;U^{-})$ are actually different. However, we remark that the product $S_{\mathbb{Z}}(x)S_{\mathbb{Z}}(-x)$ defines an entire function. □

Example 3.3 ([KuMOW]). Let $\mathbb{Q}(\tau)/\mathbb{Q}$ be an imaginary quadratic extension and $\mathbb{Z}[\tau]$ be the ring of integers of $\mathbb{Q}(\tau)$. The ring sine function $S_{\mathbb{Z}[\tau]}(x)$ of $\mathbb{Z}[\tau]$ is calculated as

$$\begin{aligned} S_{\mathbb{Z}[\tau]}(x) &:= \prod_{m,n\in\mathbb{Z}}(m+n\tau-x) \\ &= (1-e^{2\pi i x})\prod_{n=1}^{\infty}(1-e^{2\pi i(n\tau+x)})(1-e^{2\pi i(n\tau-x)}) \qquad (0<\operatorname{Im} x<\operatorname{Im}\tau)\,. \end{aligned} \tag{3.1}$$

Though it is expected from its form that the function $S_{\mathbb{Z}[\tau]}(x)$ is double-periodic, it is not. Actually, it essentially gives the elliptic theta function $\vartheta(x,\tau)$; the multi-valuedness of the attached zeta function yields the shift of the exponential factor according to the translation of the direction τ. See Figure 1. □

When one takes the linear function $\varphi(z)=z$ as in the examples above, a similar discussion as in [V], [I] assures that the function $D_{\boldsymbol{a}}(x) = \mathop{\prod\!\!\!\!\!\!\!\cdot\;\,}_n(a_n-x)$ has a Weierstrass canonical product expression. In fact, since $\mathcal{LT}$ and ∂_x is compatible (see Remark 2.3), we have the

Theorem 3.1. *Let $\boldsymbol{a}=\{a_n\}_{n\in I}$ be a sequence of non-zero complex numbers and U a connected domain in $\mathbb{C}$. Denote by p the least non-negative integer such that the series $\sum_{n\in I} a_n^{-p-1}$ converges absolutely. Suppose that the zeta function $\zeta_{\boldsymbol{a}}(s,x) := \sum_{n\in I}(a_n-x)^{-s}$ is regularizable for any $x\in U$. Then there exists a polynomial function $f_{\boldsymbol{a}}(x;U)$ of x depending on $\boldsymbol{a}$ and U such that*

$$D_{\boldsymbol{a}}(x;U) := \mathop{\prod\!\!\!\!\!\!\!\cdot\;\,}_{n\in I}(a_n-x)\Big|_U = \exp\big(f_{\boldsymbol{a}}(x;U)\big)\prod_{n\in I}\Big(1-\frac{x}{a_n}\Big)\exp\Big(\sum_{m=1}^{p}\frac{1}{m}\Big(\frac{x}{a_n}\Big)^{m}\Big)$$

holds for $x\in U$. In particular, the function $D_{\boldsymbol{a}}(x;U)$ has an analytic continuation $\tilde{D}_{\boldsymbol{a}}(x;U)$ to the whole x-plane as an entire function. □

Remark 3.1. Even if we allow the situation that the 'meromorphic' part $P(s,x;\boldsymbol{a})$ of the attached zeta function $\zeta_{\boldsymbol{a}}(s,x)$ has an essential singularity at $s=0$, the regularized product $\prod\!\!\!\!\!\!\coprod_n(a_n-x)$ is still defined and Theorem 3.1 holds. However, the function $f_{\boldsymbol{a}}(x;U)$ appearing in the theorem is no longer a polynomial function.

4 Trigonometric products

We study the zeta regularized products of trigonometric functions. In this section we establish a general theorem, and put examples and applications in the next section. For simplicity, we take $\varphi(z)=\sinh z$. Because of the differential equations

$$\begin{aligned}(\varphi')^2-\varphi^2&=1,\\ \varphi''-\varphi&=0\end{aligned}$$

satisfied by $\sinh z$, the attached zeta function satisfies a difference-differential equation which allows us to make our discussion clear.

Let $\boldsymbol{a}=\{a_n\}_{n\in I}$ be a zero-free sequence for $\varphi(z)=\sinh z$. Define the zeta function $\zeta^{\mathrm{trig}}_{\boldsymbol{a}}(s,x)$ of $\{\sinh(a_n-x)\}_{n\in I}$ by

$$\zeta^{\mathrm{trig}}_{\boldsymbol{a}}(s,x):=\sum_{n\in I}\sinh(a_n-x)^{-s}. \tag{4.1}$$

We assume that $\zeta^{\mathrm{trig}}_{\boldsymbol{a}}(s,x)$ is regularizable and holomorphic in the right half plane $\mathrm{Re}(s)>0$. Denote by μ the depth of $\zeta^{\mathrm{trig}}_{\boldsymbol{a}}(s,x)$. Since $\sinh z$ is $2\pi i$-periodic function, the function $D^{\mathrm{trig}}_{\boldsymbol{a}}(x)=\prod\!\!\!\!\!\!\coprod_n\sinh(a_n-x)$ is also a $2\pi i$-periodic function (but not entire function); we may assume that a_n's and x are lying in the strip $\mathcal{S}:=\{z\in\mathbb{C}\mid -\pi\le \mathrm{Im}\,z<\pi\}$.

The main purpose of this section is to show that the function $D^{\mathrm{trig}}_{\boldsymbol{a}}(x)$ is a **piecewise** holomorphic function, and has an analytic continuation $\tilde{D}^{\mathrm{trig}}_{\boldsymbol{a}}(x)$ as an entire function whose zeros are given by $x=a_n+k\pi i$ $(n\in I, k\in\mathbb{Z})$. More precisely, we prove the following theorem.

Theorem 4.1. *Let $\boldsymbol{a}=\{a_n\}_{n\in I}$ be a zero-free sequence for* $\sinh(z)$*, and U a connected domain in $\mathbb{C}$. Denote by p the least non-negative integer such that the series $\sum_{n\in I}a_n^{-p-1}$ converges absolutely. Suppose that the zeta function $\zeta^{\mathrm{trig}}_{\boldsymbol{a}}(s,x)=\sum_{n\in I}\sinh(a_n-x)^{-s}$ is regularizable for any $x\in U$. Put $D^{\mathrm{trig}}_{\boldsymbol{a}}(x;U):=\prod\!\!\!\!\!\!\coprod_{n\in I}\sinh(a_n-x)\big|_U$ for $x\in U$. Then there exists a polynomial function $f_{\boldsymbol{a}}(x;U)$ of x depending on $\boldsymbol{a}$ and U such that the analytic* *extension* $\tilde{D}^{\mathrm{trig}}_{\boldsymbol{a}}(x;U) = :\prod\!\!\!\!\coprod_{n\in I}\sinh(a_n - x):_U$ *of* $D^{\mathrm{trig}}_{\boldsymbol{a}}(x;U)$ *is given by*

$$\begin{aligned}\tilde{D}^{\mathrm{trig}}_{\boldsymbol{a}}(x;U) &= :\prod\!\!\!\!\coprod_{n\in I}\sinh(a_n - x):_U \\ &= \exp(f_{\boldsymbol{a}}(x;U))\prod_{\substack{n\in I\\k\in\mathbb{Z}}}\left(1-\frac{x}{a_n+k\pi i}\right)\exp\left(\sum_{m=1}^{p+1}\frac{1}{m}\left(\frac{x}{a_n+k\pi i}\right)^m\right)\end{aligned} \tag{4.2}$$

for $x\in\mathbb{C}$. *Especially, the zeros of the function* $:\prod\!\!\!\!\coprod_{n\in I}\sinh(a_n - x):_U$ *are exactly given by* $x = a_n + k\pi i$ *(*$n\in I, k\in\mathbb{Z}$*).*

Remark 4.1. When $\zeta^{\mathrm{trig}}_{\boldsymbol{a}}(s,x)$ is meromorphic at $s=0$, this result is obtained in [KiKuSW1].

Proof of Theorem 4.1. We denote by $\Delta^{\mathrm{trig}}_{\boldsymbol{a}}(x) = \Delta^{\mathrm{trig}}_{\boldsymbol{a}}(x;U)$ the right hand side of (4.2). In order to prove the equality (4.2), it is enough to show that

$$\partial_x^M \log D^{\mathrm{trig}}_{\boldsymbol{a}}(x) = \partial_x^M \log \Delta^{\mathrm{trig}}_{\boldsymbol{a}}(x) \tag{4.3}$$

for some non-negative integer $M \gg 1$ (see [V]; see also [I], [KiKuSW1]). Since $f_{\boldsymbol{a}}(x;U)$ is a polynomial function, one may suppress the symbol U in the discussion below. In fact, (4.3) implies that $\log D^{\mathrm{trig}}_{\boldsymbol{a}}(x) - \log\Delta^{\mathrm{trig}}_{\boldsymbol{a}}(x)$ is equal to a polynomial function of degree at most M.

We first note that the right hand side $\partial_x^M \log \Delta^{\mathrm{trig}}_{\boldsymbol{a}}(x)$ of (4.3) is calculated by the same discussion developed in [KiKuSW1] as follows:

Lemma 4.2. *For a sufficiently large positive integer* M*, we have*

$$\partial_x^M \log \Delta^{\mathrm{trig}}_{\boldsymbol{a}}(x) = -\eta_{\boldsymbol{a}}(M,x).$$

Here the function $\eta_{\boldsymbol{a}}(s,x)$ *is given by*

$$\eta_{\boldsymbol{a}}(s,x) := \Gamma(s)\sum_{n\in I}\sum_{k\in\mathbb{Z}}(a_n + k\pi i - x)^{-s},$$

which converges absolutely if $\mathrm{Re}(s) > p+1$. □

By the definition of the function $D^{\mathrm{trig}}_{\boldsymbol{a}}(x)$ it follows that $\log D^{\mathrm{trig}}_{\boldsymbol{a}}(x) = -\mathcal{LT}_{s=0}\,\zeta^{\mathrm{trig}}_{\boldsymbol{a}}(s,x)$. In view of the lemma above, we should hence show the equality

$$\partial_x^M \underset{s=0}{\mathcal{LT}}\,\zeta^{\mathrm{trig}}_{\boldsymbol{a}}(s,x) = \eta_{\boldsymbol{a}}(M,x) \tag{4.4}$$

for some non-negative integer $M \gg 1$. Since the operations $\mathcal{LT}$ and ∂_x commute, to show the equation (4.3) it suffices to prove the following lemma.

Lemma 4.3. *We have*

$$\underset{s=0}{\mathcal{LT}}\, \partial_x^{2N} \zeta_{\boldsymbol{a}}^{\mathrm{trig}}(s,x) = \eta_{\boldsymbol{a}}(2N,x) \tag{4.5}$$

for $N \geq \max\{\mu+2, p+2\}$. Here μ denotes the depth of the attached zeta function $\zeta_{\boldsymbol{a}}^{\mathrm{trig}}(s,x)$, and p the least non-negative integer such that the series $\sum_{n\in I} |a_n|^{-p-1}$ converges.

Before proving Lemma 4.3, we perform some preliminary calculations which make the discussion clear.

It is easy to see that

$$\partial_x^2 \sinh(a-x)^{-s} = s^2 \sinh(a-x)^{-s} + s(s+1)\sinh(a-x)^{-s-2}. \tag{4.6}$$

More generally, a successive use of the relation (4.6) leads the expression

$$\partial_x^{2N} \sinh(a-x)^{-s} = \sum_{j=0}^{N} \nu_{N,j}(s) \sinh(a-x)^{-s-2j}, \tag{4.7}$$

where $\nu_{N,j}(s)$ is a polynomial in s of degree $2N$. We note that $\nu_{N,0}(s) = s^{2N}$ and $\nu_{N,N}(s) = s(s+1)\cdots(s+2N-1)$. It follows then the

Proposition 4.4. *The zeta function $\zeta_{\boldsymbol{a}}^{\mathrm{trig}}(s,x)$ satisfies the difference-differential equation*

$$\partial_x^{2N} \zeta_{\boldsymbol{a}}^{\mathrm{trig}}(s,x) = s^{2N}\zeta_{\boldsymbol{a}}^{\mathrm{trig}}(s,x) + \sum_{j=1}^{N} \nu_{N,j}(s)\zeta_{\boldsymbol{a}}^{\mathrm{trig}}(s+2j,x) \tag{4.8}$$

for every $N \geq 1$. □

Proof of Lemma 4.3. We remark that the sum $\sum_{j=1}^{N} \nu_{N,j}(s)\zeta_{\boldsymbol{a}}^{\mathrm{trig}}(s+2j,x)$ in (4.8) is meromorphic at the origin $s=0$. Hence, if we take N so that $2N \geq \mu+2$, then the meromorphic part of $s^{2N}\zeta_{\boldsymbol{a}}^{\mathrm{trig}}(s,x)$ has a zero of order 2 at $s=0$. Thus we have

$$\underset{s=0}{\mathcal{LT}}\, \partial_x^{2N} \zeta_{\boldsymbol{a}}^{\mathrm{trig}}(s,x) = \sum_{j=1}^{N} \nu'_{N,j}(0)\zeta_{\boldsymbol{a}}^{\mathrm{trig}}(2j,x). \tag{4.9}$$

On the other hand, we notice that

$$\begin{aligned}\eta_{\boldsymbol{a}}(2N,x) &= \sum_{k\in\mathbb{Z}}\sum_{n\in I} \frac{(2N-1)!}{(a_n+k\pi i-x)^{2N}} = \sum_{n\in I} \partial_x^{2N-2} \sinh(a_n-x)^{-2} \\ &= \sum_{n\in I}\sum_{j=1}^{N} \nu_{N-1,j-1}(2)\sinh(a_n-x)^{-2j} = \sum_{j=1}^{N} \nu_{N-1,j-1}(2)\zeta_{\boldsymbol{a}}^{\mathrm{trig}}(2j,x)\end{aligned}$$

when $2N \geq p+2$. Here we use the partial fraction expansion

$$\sinh(x)^{-2} = \sum_{k\in\mathbb{Z}} \frac{1}{(x-k\pi i)^2}$$

of $\sinh(x)^{-2}$. Therefore, in order to prove Lemma 4.3, it is enough to show the equality.

$$\nu'_{N,j}(0) = \nu_{N-1,j-1}(2) \quad (j=1,2,\dots,N)\,. \tag{4.10}$$

In fact, by using the difference-differential equation (4.6) it is elementary to check that $\{\nu'_{N,j}(0)\}_{N,j}$ and $\{\nu_{N-1,j-1}(2)\}_{N,j}$ satisfy the same recurrence formula

$$\begin{aligned} &c_{N,1} = 4^{N-1}, \quad c_{N,N} = (2N-1)!,\\ &c_{N,j} = 4j^2 c_{N-1,j} - (2j-1)(2j-2)c_{N-1,j-1} \quad (1<j<N) \end{aligned}$$

as double-indexed sequences with respect to N and j. Hence (4.10) follows. This completes the proof of Lemma 4.3. □

Thus the equality (4.3) follows. This proves Theorem 4.1. □

Remark 4.2. From (4.6) we have the differential equation

$$-\partial_x^2 \log D_{\boldsymbol{a}}^{\mathrm{trig}}(x) = \underset{s=0}{\mathcal{LT}}\, s^2 \zeta_{\boldsymbol{a}}^{\mathrm{trig}}(s,x) + \sum_{n\in I} \sinh(a_n - x)^{-2} \tag{4.11}$$

of $D_{\boldsymbol{a}}^{\mathrm{trig}}(x)$. Hence, in particular there exists a polynomial function $f(x)$ such that the function $\tilde{D}_{\boldsymbol{a}}^{\mathrm{trig}}(x)$ is quasi-periodic, that is, the equality

$$\tilde{D}_{\boldsymbol{a}}^{\mathrm{trig}}(x+2\pi i) = \exp f(x) \tilde{D}_{\boldsymbol{a}}^{\mathrm{trig}}(x)$$

holds.

5 Examples and applications

5.1 q-products: $\Gamma_q(x)$ and a variant of Kronecker's limit formula

For $q>1$, put $\tau_q := \pi i/\log q$ and

$$\mathcal{S}_q := \bigl\{ z\in\mathbb{C} \bigm| -\pi/\log q \leq \operatorname{Im} z < \pi/\log q \bigr\},$$

and call it the fundamental strip of $2\tau_q$. We also put $\langle z\rangle_q := \log q^z/\log q \in \mathcal{S}_q$. Notice that $\langle z\rangle_q$ is $2\tau_q$-periodic function and $\langle z\rangle_q = z$ if $z\in\mathcal{S}_q$ (see (1.5) for the convention of the log-branch). When $q=e$, we suppress the symbol q and simply write as $\mathcal{S}$ and $\langle z\rangle$ respectively.

Since there is a basic periodicity $q^{x+2\tau_q}=q^x$, a function defined by a regularized product of q-expressions has an obvious periodicity but is not meromorphic as we show in the following examples.

Example 5.1. Let us consider the function $\coprod_{n=0}^{\infty} q^{n+x}$. The attached zeta function is

$$\zeta_{\boldsymbol{a}}(s,x)=\sum_{n=0}^{\infty}(q^{n+x})^{-s}=\frac{q^{-s\langle x\rangle_q}}{1-q^{-s}}=\frac{1}{s\log q}\sum_{n=0}^{\infty}B_n(\langle x\rangle_q)\frac{(-s\log q)^n}{n!}$$

where $B_n(t)$ denotes the n-th Bernoulli polynomial. Thus we have

$$\coprod_{n=0}^{\infty} q^{n+x}=q^{-B_2(\langle x\rangle_q)/2}\left(=q^{\zeta(-1;\langle x\rangle_q)}\right),$$

(where $B_2(x)=x^2-x+\frac{1}{6}$) which is $2\tau_q$-periodic function but not meromorphic (discontinuous on the boundary of the strip $\mathcal{S}_q$). However, the 'standard' normal product (or analytic extension)

$$\tilde{D}_{\boldsymbol{a}}(x;q^{(\,\cdot\,)};\mathcal{S}_q)=:\coprod_{n=0}^{\infty} q^{n+x}:=q^{-B_2(x)/2}$$

does not have the $2\tau_q$-periodicity. □

Example 5.2. Let $\boldsymbol{a}=\{n\}_{n=0}^{\infty}$ be the sequence of non-negative integers. Look at the function defined by

$$D_{\boldsymbol{a}}(x;[\,\cdot\,]_q):=\coprod_{n=0}^{\infty}[n+x]_q.$$

Here we denote by $[a]_q$ the q-analogue of the number a given by

$$[a]_q:=\frac{q^{a/2}-q^{-a/2}}{q^{1/2}-q^{-1/2}}.$$

If $\mathrm{Re}(a)>0$, we have

$$[a]_q^{-s}=(q^{1/2}-q^{-1/2})^s q^{-s\langle a\rangle_q}(1-q^{-a})^{-s}.$$

Using the binomial expansion we have

$$D_{\boldsymbol{a}}(x;[\,\cdot\,]_q)=\frac{[\infty]_q!}{\Gamma_q(\langle x\rangle_q)} \tag{5.1}$$

for $\mathrm{Re}(x)>0$ (see [KuW2]). Here $\Gamma_q(x)$ denotes the Jackson q-gamma function defined by

$$\Gamma_q(x):=\frac{\prod_{n=1}^{\infty}(1-q^{-n})}{\prod_{n=0}^{\infty}(1-q^{-(n+x)})}(q^{1/2}-q^{-1/2})^{1-x}q^{x(x-1)/4},$$

and the constant $[\infty]_q! := \prod_{n=0}^{\infty}[n]_q$ is essentially the Dedekind η-function given by

$$[\infty]_q! = (q^{1/2} - q^{-1/2})^{-\log(1-q^{-1})/\log q} \times q^{-1/24} \prod_{n=1}^{\infty}(1 - q^{-n}).$$

Because of the functional equation $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$, (5.1) is valid for all $x \in \mathbb{C}$ by virtue of the property $\prod_{n\in I} a_n \prod_{n\in J} a_n = \prod_{n\in I\sqcup J} a_n$ (see (2.2)). The function $D_{\boldsymbol{a}}(x;[\,\cdot\,]_q)$ is holomorphic in each strip but is not an entire function. We also notice that an analytic extension (the 'standard' normal product)

$$\tilde{D}_{\boldsymbol{a}}(x;[\,\cdot\,]_q;\mathcal{S}_q) = :\prod_{n=0}^{\infty}[n+x]_q: = \frac{[\infty]_q!}{\Gamma_q(x)}$$

of $D_{\boldsymbol{a}}(x;[\,\cdot\,]_q)$ becomes entire, but the periodicity is not preserved; still $\Gamma_q(x)$ has a quasi-periodicity

$$\Gamma_q(x + 2\tau_q) = (q^{1/2} - q^{-1/2})^{2\tau_q} q^{\tau_q x + \tau_q^2 - 1/2}\Gamma_q(x).$$

As we have seen in Section 3, this kind of quasi-periodicity is inherited from the existence of differential equations of exponential (and/or trigonometric) functions. See Remark 4.2.

We put $U_k := \mathcal{S}_q + 2k\tau_q$ for $k \in \mathbb{Z}$.

$$D_{\boldsymbol{a}}(x;[\,\cdot\,]_q;U_k) = \frac{[\infty]_q!}{\Gamma_q(x)} F_q(x;U_k) \qquad (x \in U_k)$$

where

$$F_q(x;U_k) := (q^{1/2} - q^{-1/2})^{-2k\tau_q} q^{k\tau_q x - k\tau_q(k\tau_q + 1/2)} \qquad (k \in \mathbb{Z})$$

Thus, two entire functions $\tilde{D}_{\boldsymbol{a}}(x;[\,\cdot\,]_q;U_k)$ and $\tilde{D}_{\boldsymbol{a}}(x;[\,\cdot\,]_q;U_m)$ coincide up to the factor $F_q(x;k)F_q(x;m)^{-1}$. □

Example 5.3. A q-analogue $S^q_{\mathbb{Z}}(x) := :\prod_{n\in\mathbb{Z}}[n-x]_q:$ of the ring sine function $S_{\mathbb{Z}}(x)$ substantially gives the elliptic theta function $\vartheta(x) = \vartheta(x; \frac{\log q}{2\pi i})$. Actually, we have

$$S^q_{\mathbb{Z}}(x) = (\text{a constant}) \times \frac{[\infty]_q!^2}{\Gamma_q(x)\Gamma_q(1-x)}.$$

This is a variant of Kronecker's limit formula. In fact, $S^q_{\mathbb{Z}}(x/\tau)$ is essentially equal to $S_{\mathbb{Z}[\tau]}(x)$ in Example 3.3 (see [KiKuSW1, Remark 4.4]). See also Remark 5.4 for a comparison of $|S^q_{\mathbb{Z}}(x)|$ with $\prod_{n\in\mathbb{Z}}|[n-x]_q|$. □

5.2 Products over the essential zeros of $\zeta(s)$ and the RH

The higher Riemann zeta function $\zeta_{l\infty}(s)$ $(l = 1, 2, \dots)$ is defined by

$$\zeta_{l\infty}(s) := \prod_{n=0}^{\infty} \zeta(s + ln).$$

This $\zeta_{l\infty}(s)$ is analytically continued to the whole plane as a meromorphic function. As the Riemann zeta function $\zeta(s)$ possesses a functional equation, the higher Riemann zeta function $\zeta_{l\infty}(s)$ also has a functional equation between s and $1 - s - l$ [KuMW]. If one hopes to write this functional equation in a symmetric form, then, beside the gamma factor described by $\Gamma(s)$ and the double gamma function $\Gamma_2(s)$ (see (6.1) for the definition), it is necessary to introduce the function $S_\alpha(x)$ of the form

$$S_\alpha(x) := \underset{\operatorname{Im}\rho>0}{\prod\nolimits^{\star}} \sin\alpha(\rho - x) \qquad (\alpha > 0)$$

where $\prod^{\star}$ denotes a regularized product in a suitable sense and ρ runs through the essential zeros of $\zeta(s)$ with positive imaginary part. The initial purpose of this subsection is to show that the function $S_\alpha(x)$ exists if we take $\prod^{\star} = \prod^{\vdots}$. In fact, we find a certain expression of $S_\alpha(x)$ by a usual infinite product of the factors $1 - e^{2\alpha i\rho x}$. Consequently, we may construct the function $G_{l\infty}(s)$ explicitly in terms of $\Gamma(s)$, $\Gamma_2(s)$ and $S_{\pi/l}(s)$ so that the completion $\hat{\zeta}_{l\infty}(s) := G_{l\infty}(s)\zeta_{l\infty}(s)$ of $\zeta_{l\infty}(s)$ satisfies the symmetric functional equation

$$\hat{\zeta}_{l\infty}(s)\hat{\zeta}_{l\infty}(1 - s - l) = 1.$$

Moreover, using the ddotted product representation of $S_\alpha(x)$, as an application we establish a certain statement which is equivalent to the validity of some distribution formula of the essential zeros of $\zeta(s)$ in Theorem 5.4; This is regarded as a weaker version of Riemann Hypothesis of $\zeta(s)$.

Recall first the Cramér's result [Cr]. Let us consider the following partition functions

$$V(w) := \sum_{\operatorname{Im}\rho>0} e^{\rho w} \qquad (\operatorname{Im}(w) > 0)\,,$$

$$\Phi(t) := \sum_{\operatorname{Re}\tau>0} e^{-\tau t} \qquad (\operatorname{Re}(t) > 0)\,.$$

Here the summation $\sum_{\operatorname{Re}\tau>0}$ is taken over all essential zeros ρ of $\zeta(s)$ such that $\operatorname{Re}(\tau) > 0$ where $\tau = \tau(\rho)$ is defined by $\rho = 1/2 + i\tau$. It is easy to see that $V(it) = e^{it/2}\Phi(t)$ for $\operatorname{Re}(t) > 0$. It is proved in [Cr] that the function

$$V(w) - \frac{1}{2\pi i}\left(\frac{\log w}{1 - e^{-w}} + \frac{\gamma + \log 2\pi - \pi i/2}{w}\right) \tag{5.2}$$

is analytically continued to the whole w-plane as a single-valued function, and is holomorphic near the origin $w = 0$. Here γ denotes the Euler constant. More precisely, we have the

Lemma 5.1. *For any $\alpha > 0$, the meromorphic part $\varphi_\alpha^{\mathrm{mero}}(s)$ of $\Phi(\alpha s)$ is given by*

$$\varphi_\alpha^{\mathrm{mero}}(s) = \lambda_{-1}(\alpha)s^{-1} + \lambda_0(\alpha) + \lambda_1(\alpha)s + O(s^2)$$

where the coefficients λ_{-1} and λ_0 are explicitly given as follows:

$$\lambda_{-1}(\alpha) = -\frac{\gamma + \log 2\pi\alpha}{2\pi\alpha}, \quad \lambda_0(\alpha) = \frac{7}{8}.$$

□

Using this lemma we remark first the following simple example of the ddotted product.

Example 5.4. The ddotted regularized product

$$\mathop{\boxed{:}}_{\mathrm{Im}\,\rho>0} e^{i(x-\rho)} = \exp\left(\frac{1}{2}\left(x - \frac{1}{2}\right)^2 \frac{\gamma + \log 2\pi}{2\pi} - \frac{7i}{8}x + C\right)$$

is obtained from the differential equation $\partial_x T(s,x) = -isT(s,x)$ of the attached zeta function

$$T(s,x) := \sum_{\mathrm{Im}\,\rho>0} e^{-is(x-\rho)} = e^{(1/2-x)is}\Phi(s).$$

Here C is some constant. □

To study the function $S_\alpha(x)$, let us calculate the attached zeta function

$$L_\alpha(s,x) := \sum_{\mathrm{Im}\,\rho>0} \sin\alpha(\rho - x)^{-s}$$

for observing the existence of the function $S_\alpha(x)$. Suppose that $\mathrm{Im}\,x \leq 0$. Using the binomial theorem we see that

$$\begin{aligned}
L_\alpha(s,x) &= \sum_{\mathrm{Im}\,\rho>0} \left(\frac{e^{i\alpha(\rho-x)} - e^{-i\alpha(\rho-x)}}{2i}\right)^{-s} \\
&= (-2i)^s \sum_{\mathrm{Im}\,\rho>0} e^{i\alpha s(\rho-x)}(1 - e^{2i\alpha(\rho-x)})^{-s} \\
&= e^{f_\alpha(x)s} \sum_{\mathrm{Im}\,\rho>0} e^{i\alpha s\rho} \sum_{n=0}^{\infty} \binom{-s}{n} (-1)^n e^{2ni\alpha(\rho-x)} \\
&= e^{f_\alpha(x)s} \left\{ V(i\alpha s) + \sum_{n=1}^{\infty} (-1)^n \binom{-s}{n} V(i\alpha(s+2n)) e^{-2ni\alpha x} \right\}
\end{aligned}$$

where we put $f_\alpha(x) := -i\alpha x + \log(-2i)$. Since the function $V(i\alpha(s+2n))$ is holomorphic at $s = 0$, we obtain

$$(-1)^n \binom{-s}{n} e^{-2ni\alpha x} V(i\alpha(s+2n)) = \frac{V(2i\alpha n)}{n} e^{-2ni\alpha x} s + O(s^2)$$

around $s = 0$. Therefore we have

$$L_\alpha(s,x) = e^{f_\alpha(x)s} \left\{ e^{i\alpha s/2} \Phi(\alpha s) + s P_\alpha(x) + O(s^2) \right\}. \tag{5.3}$$

Here we put

$$P_\alpha(x) := \sum_{n=1}^{\infty} \frac{V(2\alpha n)}{n} e^{-2ni\alpha x}.$$

This shows that $L_\alpha(s,x)$ is regularizable by (5.2). Moreover, the linear term of $L_\alpha(s,x)$ is given by

$$\begin{aligned} \mathop{\mathcal{LT}}_{s=0} L_\alpha(s,x) &= \mathop{\mathcal{LT}}_{s=0} \left\{ e^{s(f_\alpha(x) + i\alpha/2)} \Phi(\alpha s) \right\} + P_\alpha(x) \\ &= F_\alpha(x) + P_\alpha(x). \end{aligned} \tag{5.4}$$

Here $F_\alpha(x)$ is a quadratic polynomial $F_\alpha(x) = A_\alpha x^2 + B_\alpha x + C_\alpha$, where the coefficients A_α and B_α are explicitly given by

$$A_\alpha = \frac{\alpha(\gamma + \log 2\pi\alpha)}{4\pi}, \quad B_\alpha = -i\alpha \left(\frac{\gamma + \log 2\pi\alpha}{2\pi} \left(\frac{\log(-2i)}{\alpha} + \frac{i}{2} \right) - \frac{7}{8} \right).$$

Consequently, we obtain the following infinite product expression of $S_\alpha(x)$.

Theorem 5.2. *The function* $S_\alpha(x) := :\boxed{\vdots}_{\operatorname{Im}\rho>0} \sin\alpha(\rho - x):$ *exists. Here the initial domain of this normal product is taken as* $\left\{ x \in \mathbb{C} \;\middle|\; 0 \le \operatorname{Re} x < 2\pi/\alpha \right\}$. *It also has the product expression*

$$\begin{aligned} S_\alpha(x) &= e^{-F_\alpha(x)} \prod_{\operatorname{Im}\rho>0} (\sin\alpha(\rho - x))\, e^{i\alpha(\rho - x) + \log(-2i)} \\ &= e^{-F_\alpha(x)} (e^{-2\alpha i x}; e^{-2\alpha i})_\zeta. \end{aligned} \tag{5.5}$$

Here we put

$$(x;q)_\zeta := \prod_{\operatorname{Im}\rho>0} (1 - x q^{-\rho}).$$

Proof. Since two functions $S_\alpha(x)$ and $(e^{-2\alpha i x}; e^{-2\alpha i})_\zeta$ have the same zeros and are of order 2, they coincide up to a quadratic factor, that is, there exists a certain quadratic polynomial

$g_\alpha(x)$ such that $S_\alpha(x) = e^{g_\alpha(x)}(e^{-2\alpha ix}; e^{-2\alpha i})_\zeta$. By taking the logarithm in the initial domain we observe that

$$-\log S_\alpha(x) = \underset{s=0}{\mathcal{LT}}\, L_\alpha(s,x) = F_\alpha(x) + P_\alpha(x) = -g_\alpha(x) - \log(e^{-2\alpha ix}; e^{-2\alpha i})_\zeta.$$

When x tends to $-i\infty$ along the imaginary axis, the functions $P_\alpha(x)$ and $-\log(e^{-2\alpha ix}; e^{-2\alpha i})_\zeta$ vanish. Therefore, two functions $-F_\alpha(x)$ and $g_\alpha(x)$ must coincide since they are polynomial functions. This completes the proof. □

We notice that the function $\Phi(t)$ is real-valued if t is real. Actually, the function $\Phi(t)$ has the expression

$$\Phi(t) = 2\sum_{\substack{\mathrm{Re}\,\tau>0\\ \mathrm{Im}\,\tau>0}} e^{-t\,\mathrm{Re}\,\tau}\cos(t\,\mathrm{Im}\,\tau) + \sum_{\substack{\tau\in\mathbb{R}\\ \tau>0}} e^{-t\tau}. \tag{5.6}$$

If we introduce the functions

$$\Phi^R(t) := \sum_{\mathrm{Re}\,\tau>0} e^{-t\,\mathrm{Re}\,\tau},$$

$$\Psi(t) := \sum_{\substack{\mathrm{Re}\,\tau>0\\ \mathrm{Im}\,\tau>0}} e^{-t\,\mathrm{Re}\,\tau}(\mathrm{Im}\,\tau)^2\left\{\frac{\sin(t\,\mathrm{Im}\,\tau/2)}{t\,\mathrm{Im}\,\tau/2}\right\}^2,$$

then the functions $\Phi(t)$, $\Phi^R(t)$ and $\Psi(t)$ satisfies the following relations.

Lemma 5.3. *For sufficiently small $t > 0$, we have*

$$0 \le \Phi^R(t) - \Phi(t) = t^2\Psi(t) \le \frac{t^2/8}{1-t^2/8}\Phi(t). \tag{5.7}$$

Proof. Remark that $|\mathrm{Im}\,\tau| < 1/2$ since $\rho = 1/2 + i\tau$ lies in the critical strip $0 < \mathrm{Re}(\rho) < 1$. Then we have

$$\begin{aligned}
\Phi^R(t) - \Phi(t) &= 2\sum_{\substack{\mathrm{Re}\,\tau>0\\ \mathrm{Im}\,\tau>0}} e^{-t\,\mathrm{Re}\,\tau}\left(1-\cos(t\,\mathrm{Im}\,\tau)\right)\\
&= 4\sum_{\substack{\mathrm{Re}\,\tau>0\\ \mathrm{Im}\,\tau>0}} e^{-t\,\mathrm{Re}\,\tau}\sin^2(t\,\mathrm{Im}\,\tau/2)\\
&= t^2\sum_{\substack{\mathrm{Re}\,\tau>0\\ \mathrm{Im}\,\tau>0}} e^{-t\,\mathrm{Re}\,\tau}(\mathrm{Im}\,\tau)^2\left\{\frac{\sin(t\,\mathrm{Im}\,\tau/2)}{t\,\mathrm{Im}\,\tau/2}\right\}^2 \left(=t^2\Psi(t)\right)\\
&\le \frac{t^2}{4}\sum_{\substack{\mathrm{Re}\,\tau>0\\ \mathrm{Im}\,\tau>0}} e^{-t\,\mathrm{Re}\,\tau} \le \frac{t^2}{8}\Phi^R(t),
\end{aligned}$$

from which the desired inequalities are immediately obtained. □

Theorem 5.4. *Define a quadratic polynomial $R_\alpha(u)$ by*

$$R_\alpha(u) := -\frac{\gamma + \log 2\pi\alpha}{4\pi\alpha}\left\{\alpha\left(u - \frac{1}{2}\right) + \frac{\pi}{2}\right\}^2.$$

Suppose that the regularized product $\prod_{\operatorname{Im}\rho>0} |\sin\alpha(\rho - x)|$ *exists. Then, the following two conditions are equivalent.*

(i) The equality

$$\prod_{\operatorname{Im}\rho>0} \big|\sin\alpha(\rho - x)\big| = e^{-R_\alpha(\operatorname{Re}(x))}\left|\prod_{\operatorname{Im}\rho>0} \sin\alpha(\rho - x)\right| \qquad (\operatorname{Im} x \le 0) \tag{5.8}$$

holds for two distinct values of $\alpha > 0$.

(ii) An asymptotic formula $\sum_{\operatorname{Im}\rho>0}(\operatorname{Re}\rho - 1/2)^2 e^{-t\operatorname{Im}\rho} = O(\log t)$ holds as $t \to 0$.

Proof. Denote by $\tilde{L}_\alpha(s,x)$ the attached zeta function

$$\tilde{L}_\alpha(s,x) := \sum_{\operatorname{Im}\rho>0} \big|\sin\alpha(\rho - x)\big|^{-s}$$

of the regularized product $\prod_{\operatorname{Im}\rho>0} |\sin\alpha(\rho - x)|$. First we see that

$$\tilde{L}_\alpha(s,x) = e^{s\tilde{f}_\alpha(x)} \sum_{\operatorname{Im}\rho>0} e^{-s\alpha\operatorname{Im}(\rho)}\big|1 - e^{2i\alpha(\rho - x)}\big|^{-s}$$

where we put $\tilde{f}_\alpha(x) := \alpha\operatorname{Im}(x) + \log 2 = \operatorname{Re} f_\alpha(x)$. Since

$$\begin{aligned}
\big|1 - e^{2i\alpha(\rho-x)}\big|^{-s} &= \big(1 - e^{2i\alpha(\rho-x)}\big)^{-s/2}\big(1 - e^{-2i\alpha(\bar\rho-\bar x)}\big)^{-s/2} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{-s/2}{m}\binom{-s/2}{n}(-1)^m e^{2mi\alpha(\rho-x)}(-1)^n e^{-2ni\alpha(\bar\rho-\bar x)} \\
&= 1 + \frac{s}{2}\sum_{m=1}^{\infty}\frac{1}{m}e^{2mi\alpha(\rho-x)} + \frac{s}{2}\sum_{n=1}^{\infty}\frac{1}{n}e^{-2ni\alpha(\bar\rho-\bar x)} + O(s^2),
\end{aligned}$$

we have

$$\begin{aligned}
&\sum_{\operatorname{Im}\rho>0} e^{-s\alpha\operatorname{Im}(\rho)}\big|1 - e^{2i\alpha(\rho-x)}\big|^{-s} \\
&= \sum_{\operatorname{Im}\rho>0} e^{-s\alpha\operatorname{Im}\rho} + \frac{s}{2}\sum_{m=1}^{\infty}\frac{e^{-2mi\alpha x}}{m}\sum_{\operatorname{Im}\rho>0} e^{2mi\alpha\rho} + \frac{s}{2}\sum_{n=1}^{\infty}\frac{e^{2ni\alpha\bar x}}{n}\sum_{\operatorname{Im}\rho>0} e^{-2ni\alpha\bar\rho} + O(s^2) \\
&= \Phi^R(s\alpha) + s\operatorname{Re} P_\alpha(x) + O(s^2).
\end{aligned} \tag{5.9}$$

Hence we have

$$\tilde{L}_\alpha(s,x) = e^{s\tilde{f}_\alpha(x)}\left(\Phi^R(s\alpha) + s\,\mathrm{Re}\,P_\alpha(x) + O(s^2)\right). \tag{5.10}$$

Assume that the regularized product $∏_{\mathrm{Im}\,\rho>0}|\sin\alpha(\rho-x)|$ exists. Then the function $\Psi(t)$ is written in the form $\Psi(t) = P(t) + \sum_{j=1}^m Q_j(t)(\log t)^m$ for some meromorphic functions $Q_j(t)$. Thanks to the inequality (5.7), it is elementary to check that $m=1$, that is, $\Psi(t)$ is in the form

$$\Psi(t) = P(t) + Q(t)\log t \tag{5.11}$$

and $P(t)$, $Q(t)$ have at most simple poles at $t=0$.

By (5.3) and (5.10) we have

$$\begin{aligned}
&\mathop{\mathcal{LT}}_{s=0}\tilde{L}_\alpha(s,x) - \mathrm{Re}\mathop{\mathcal{LT}}_{s=0}L_\alpha(s,x)\\
=&\mathop{\mathcal{LT}}_{s=0}\left\{e^{s\tilde{f}_\alpha(x)}\Phi(s\alpha)\right\} - \mathrm{Re}\mathop{\mathcal{LT}}_{s=0}\left\{e^{s(f_\alpha(x)+i\alpha/2)}\Phi(s\alpha)\right\} + \mathop{\mathcal{LT}}_{s=0}e^{s\tilde{f}_\alpha(x)}\left\{\Phi^R(s\alpha)-\Phi(s\alpha)\right\}\\
=&R_\alpha(\mathrm{Re}(x)) + \alpha^2\mathop{\mathcal{LT}}_{s=0}s^2e^{s\tilde{f}_\alpha(x)}\Psi(s\alpha)
\end{aligned}$$

Hence the validity of (5.8) is equivalent to the condition $\mathcal{LT}_{s=0}\,s^2e^{s\tilde{f}_\alpha(x)}\Psi(s\alpha) = 0$. Since $P(s\alpha)$ and $Q(s\alpha)$ have at most simple poles at $s=0$, we see that

$$\begin{aligned}
&\mathop{\mathcal{LT}}_{s=0}s^2e^{s\tilde{f}_\alpha(x)}\Psi(s\alpha)\\
=&\mathop{\mathcal{LT}}_{s=0}\left(s^2(1+s\tilde{f}_\alpha(x)+s^2\tilde{f}_\alpha(x)^2+\cdots)(P(s\alpha)+Q(s\alpha)\log\alpha+Q(s\alpha)\log s)\right)\\
=&\mathop{\mathrm{Res}}_{s=0}P(s\alpha) + \log\alpha\mathop{\mathrm{Res}}_{s=0}Q(s\alpha).
\end{aligned}$$

By taking two distinct values of α, this implies that $\mathcal{LT}_{s=0}\,s^2e^{s\tilde{f}_\alpha(x)}\Psi(s\alpha) = 0$ if and only if $\mathrm{Res}_{s=0}P(s) = \mathrm{Res}_{s=0}Q(s) = 0$. which is also equivalent to the estimation $\Psi(t) = O(\log t)$ as $t\to 0$ in view of (5.11). If we recall the asymptotics $\Psi(t)\sim\sum_{\mathrm{Re}\,\tau>0}(\mathrm{Im}\,\tau)^2e^{-\mathrm{Re}\,\tau t}$, which is immediate from the definition of $\Psi(t)$, the assertion of the theorem is now clear. □

Remark 5.1. It is interesting to study the convergence of the series

$$\sum_{\mathrm{Re}\,\tau>0}(\mathrm{Re}\,\tau)^x(\mathrm{Im}\,\tau)^2 \quad (x\geq 0).$$

We note that if the series above converges for every $x>0$, then the existence of the regularized product $∏_\rho|\sin\alpha(\rho-x)|$ follows.

Remark 5.2. In view of Theorem 5.2, it would be very interesting to examine if the ddotted product $\ddot{\prod}_\rho |\sin\alpha(\rho-x)|$ possesses an infinite product expression such as $:\ddot{\prod}_\rho \sin\alpha(\rho-x):$ has. See (5.5).

Remark 5.3. We can extend the definition of regularized products as follows: Suppose that the attached zeta function $\zeta_{\boldsymbol{a}}(s)$ of a given sequence $\boldsymbol{a}=\{a_n\}_n$ is analytically continued to the domain $\mathrm{Re}(s)>0$. Denote by $\varphi(s)$ the meromorphic part of $\zeta_{\boldsymbol{a}}(s)$. Namely, there exist a finite number of functions $\beta_j(s)$, $f_j(s)$ such that $\beta_j(s)$ is not meromorphic (has a log singularity or a branch point) at $s=0$, $f_j(s)$ is meromorphic at $s=0$ and satisfies $\zeta_{\boldsymbol{a}}(s)=\varphi(s)+\sum_j \beta_j(s)f_j(s)$. Then we define

$$\ddot{\prod_n} a_n := \exp\left(-\operatorname*{Res}_{s=0}\frac{\varphi(s)}{s^2}\right).$$

It can be proved that a function defined by this regularized product also has the Wierstrass canonical form as in Theorems 3.1 and 4.1. If we adopt this definition, then the condition corresponding to (ii) in the theorem above varies in delicate way.

Remark 5.4. The analogous formulas of (5.5) hold for the q-gamma function $\Gamma_q(x)$ and the q-analogue of the ring sine function $S_{\mathbb{Z}}^q(x)$ defined in Example 5.3. In fact, we have

$$\dot{\prod_{n\geq 0}}\big|[n-x]_q\big| = q^{(x-\bar{x})^2/16}\times\left|\dot{\prod_{n\geq 0}}[n-x]_q\right|, \tag{5.12}$$

$$\dot{\prod_{n\in\mathbb{Z}}}\big|[n-x]_q\big| = e^{-\pi^2/\log q}q^{(x-\bar{x})^2/8}\left|\dot{\prod_{n\in\mathbb{Z}}}[n-x]_q\right|. \tag{5.13}$$

We note that the left hand sides in these formulas are also directly calculated by definition in contrast with $\ddot{\prod}_{\mathrm{Im}\,\rho>0}|\sinh(\rho-x)|$. We remark further that the exponential factors appearing in (5.12) and (5.13) is a kind of an anomaly in view of (2.5) (see Remark 2.2). It would be also interesting to give an interpretation of these exponential factors in a geometric way, for instance, by some intersection numbers.

Remark 5.5. It is hard to establish a criterion similar to the one in Theorem 5.4 about the relation between the half zeta function $\zeta^+(s) := \dot{\prod}_{\mathrm{Im}\,\rho>0}(\rho-x)$ (see [HKuW]) and $\dot{\prod}_{\mathrm{Im}\,\rho>0}|\rho-x|$.

5.3 Products over the eigenvalues of the Laplacian Δ_Γ

A similar situation occurs when we study the symmetric functional equation of the higher Selberg zeta function. We recall the Selberg zeta function $Z_\Gamma(s)$ defined by

$$Z_\Gamma(s) := \prod_{m=1}^{\infty} \prod_{P \in \mathrm{Prim}(\Gamma)} (1 - N(P)^{-s-m}) \qquad (\mathrm{Re}(s) > 1)$$

for a (uniform or non-uniform) lattice Γ in $PSL(2, \mathbb{R})$. Here $\mathrm{Prim}(\Gamma)$ denotes the set of primitive hyperbolic conjugacy classes of Γ, and $N(P) := \max\{|\alpha_P|^2, |\beta_P|^2\}$ is the norm of P where α_P and β_P are the eigenvalues of a representative matrix of P. The higher Selberg zeta function $z_\Gamma(s)$ of Γ is also defined by

$$z_\Gamma(s) := \prod_{m=1}^{\infty} \prod_{P \in \mathrm{Prim}(\Gamma)} (1 - N(P)^{-s-m})^{-m} = \prod_{n=1}^{\infty} Z_\Gamma(s+n)^{-1}.$$

Both $z_\Gamma(s)$ and $Z_\Gamma(s)$ are meromorphic in the entire plane [KuW3]. Related to the study of the symmetric functional equation of $z_\Gamma(s)$, it is useful to introduce the function of the form

$$\text{``}\prod_{n \geq 0}^{\star} \cosh(r_n - x)\text{''}$$

where r_n is a normalized eigenvalues of the Laplacian Δ_Γ on $L^2(\Gamma\backslash H)$, that is, the discrete spectrum of Δ_Γ is given by $\mathrm{Spec}\,\Delta_\Gamma = \{\lambda_n = 1/4 + r_n^2\}_{n \geq 0}$ $(0 = \lambda_0 \leq \lambda_1 \leq \dots)$. Here the eigenvalue $\lambda_0 = 1/4 + r_0^2 = 0$ (i.e. $r_0 = -i/2$) corresponds to the space of constant functions on $\Gamma\backslash H$. We take r_n such as $\mathrm{Re}(r_n) > 0$. When Γ is a uniform lattice (i.e. Γ is co-compact), the dotted product $\prod^{\cdot}_{n \geq 0} \cosh(r_n - x)$ exists (see [KuW3]). If we take $\Gamma = \Gamma_i(N)$ a congruence subgroup of $PSL(2, \mathbb{R})$ (which is non-uniform), it is shown that the situation is the same as that of Cramér's $V(w)$ above and we actually need the ddotted product. Define the theta function

$$\Theta_\Gamma(t) := \sum_{n=0}^{\infty} e^{-tr_n}. \qquad (t > 0)$$

Then, there exists some constant C such that the function $\Theta_\Gamma(t) - C \log t$ is analytically continued to the whole t-plane by virtue of a Cartier-Voros type trace formula for Γ (see [CaV], [H2]). Thus, by the same discussion as in the proof of Theorem 5.2, we have the

Theorem 5.5. *For a congruence subgroup Γ of $PSL(2, \mathbb{R})$, the ddotted regularized product $\prod^{:}_{n \geq 0} \cosh(r_n - x)$ exists and can be extended as an entire function. Indeed, there exists a* *polynomial* $f_\Gamma(x)$ *of degree* 3 *such that*

$$\det\cosh\left(\sqrt{\Delta_\Gamma-\frac{1}{4}}-x\right):=:\prod_{n\geq 0}\cosh(r_n-x):=e^{f_\Gamma(x)}\prod_{n\geq 0}(1+e^{-2(r_n-x)}).$$

Here the initial domain of this normal product is taken as $\{z\in\mathbb{C}\mid 0\leq \operatorname{Im}x<2\pi\}$*.* □

Remark 5.6. When Γ is co-compact, it is known in [KuW3] that the polynomial $f_\Gamma(x)$ can be written as

$$f_\Gamma(x)=-\frac{1}{3\pi^2}(g-1)(i\pi x+\log 2)^3+\frac{1}{12}(g-1)(i\pi x+\log 2)-b_1$$

for some constant b_1. Here g denotes the genus of the Riemann surface $\Gamma\backslash H$.

Remark 5.7. Let Γ be a congruence subgroup of $PSL(2,\mathbb{R})$. It is shown in [H1] that the function $\Theta_\Gamma(t)$ is given by the form

$$\Theta_\Gamma(t)=Q_\Gamma(t)\log t+\frac{A_\Gamma}{t}+B_\Gamma+C_\Gamma t+O(t^2).$$

Here the coefficients are given explicitly. We note that if we put $R^\Gamma_\alpha(x)=A_\Gamma\alpha\operatorname{Im}(x)^2$, then the identity

$$\prod_{n\geq 0}\left|\cosh\alpha(r_n-x)\right|=e^{-R^\Gamma_\alpha(x)}\left|\prod_{n\geq 0}\cosh\alpha(r_n-x)\right| \tag{5.14}$$

holds for some $\alpha>0$ if and only if Γ satisfies Selberg's 1/4-conjecture for the first eigenvalue of the Laplacian Δ_Γ (see, e.g. [S]) which implies that there is no exceptional zeros (i.e. $\lambda_1=1/4+r_1^2\geq 1/4$ for $n\geq 1$) of the Selberg zeta function $Z_\Gamma(s)$. This is proved by the same way in Theorem 5.4. Since it is known that the Selberg zeta function $Z_\Gamma(s)$ for the modular group $\Gamma=SL(2,\mathbb{Z})$, for instance, satisfies an analogue of the Riemann Hypothesis (i.e. satisfies the Selberg's 1/4-conjecture), we have the identity (5.14).

5.4 Necessary regularization

For a given function φ, we look at the following three kinds of regularized products.

(Linear) $\prod^{\star}_{\rho}(\rho-x)$,

(Half) $\prod^{\star}_{\operatorname{Im}\rho>0}(\rho-x)$,

(Trigonometric) $\prod^{\star}_{\operatorname{Im}\rho>0}\sinh(\rho-x)$.

Here the product symbol $\prod^{\star}$ indicates a suitable regularized product, and ρ runs through the (normalized) non-trivial zeros of either the Selberg zeta functions $Z_\Gamma(s)$ for a uniform lattice, $Z_\Gamma(s)$ for a non-uniform lattice or the Riemann zeta function $\zeta(s)$. Table 1 shows the necessary regularization for these three products (see also Examples 6.2 and 6.3).

	compact $Z_\Gamma(s)$	noncompact $Z_\Gamma(s)$	$\zeta(s)$
Linear	$\prod$	$\prod$	$\prod$
Half	$\prod$	$\prod^{\bullet}$	$\prod^{\bullet}$
Trigonometric	$\prod^{\bullet}$	$\prod^{\bullet\bullet}$	$\prod^{\bullet\bullet}$

Table 1: Necessary regularization

6 Concluding remarks

As final remarks, we describe the hierarchy of regularizations first, some experimental observation concerning the theta functions, Jacobi forms second, and about the zeta extensions related to the aforementioned examples in Section 5.

6.1 Hierarchy of regularizations

Table 2 summarizes the resulting functions of regularized products of the form

$$\prod_{n\in L}^{\star} \varphi(n-x)$$

where L is a semi-lattice or lattice, $\varphi(z)$ is one of the functions z, $\sinh z$, $\theta(z)$.

In Table 2, $\Gamma_n(x)$ is the Barnes multiple gamma function (see, e.g. [B], [KuKo]) defined by

$$\frac{1}{\Gamma_n(x)} := \prod_{k_1,\dots,k_n\geq 0} (k_1+\cdots+k_n+x), \tag{6.1}$$

and $O_q(x;n)$ is the Appel O-function of rank n (see, e.g. [KuW2]) defined by

$$O_q(x;n) := \prod_{k_1,\dots,k_n\geq 0} \left(1-q^{-(k_1+\cdots+k_n+x)}\right).$$

	Rational $\prod$	Trigonometric $\boxdot$	Elliptic $\boxed{\star}$
$\mathbb{Z}_{\geq 0}$	$\Gamma(x)$	$\Gamma_q(x)$	$\mathbf{?}^1$
$\mathbb{Z}$	$\sin x$	$\vartheta(x;\tau)$	"$\vartheta(x;\tau)$"
$\mathbb{Z}_{\geq 0}^{\oplus n}$	$\Gamma_n(x)$	$O_q(x;n)$	$\mathbf{?}^1$
$\mathbb{Z}^{\oplus 2}$	$\vartheta(x)$	$\mathbf{?}^2$	$\mathbf{?}^2$

Table 2: Knight moving table

For the theta function "$\vartheta(x;\tau)$" appearing in the elliptic column of Table 2, see Section 6.2. We have not succeeded in obtaining regularized products corresponding to $\mathbf{?}^1$ and $\mathbf{?}^2$ in Table 2 yet. The difficulty in $\mathbf{?}^1$ is lied in the analysis of the behavior of the attached zeta function at the origin $s = 0$. In the case of $\mathbf{?}^2$, even the attached zeta function does not exist in the present sense. Thus, for example, the ring sine function of the integer ring of a real quadratic field cannot be defined via the regularized product.

We hope that there exist a transitive relation such as

$$\mathop{\boxed{\star}}_{m\in I}\mathop{\boxed{\star}}_{n\in J} f(x;m,n) = \mathop{\boxed{\star}}_{\substack{m\in I\\ n\in J}} f(x;m,n).$$

From this viewpoint we may expect the presence of a hierarchy

$$\mathop{\boxed{\star}}_{n\geq 0} \Gamma_k(n+x) = \Gamma_{k+1}(x)$$

among the multiple gamma functions under a suitable formulation of zeta regularized products. Related to this expectation, see Example 6.1 in Section 6.3.

6.2 Towards the elliptic products

6.2.1 Elliptic theta function $\vartheta(x,t)$

We present here an experimental study towards a possibility to defining a regularized product of the elliptic theta functions

$$\Theta(x,t) = \mathop{\boxed{\star}}_{k\in\mathbb{Z}} \vartheta(x+kt,t). \tag{6.2}$$

We are interested in what kind of a new regularization we should employ for $\boxed{\star}$ in (6.2). Recall the theta function $\vartheta(x,t)$ is defined by

$$\vartheta(x,t) := \sum_{n\in\mathbb{Z}} \exp(-n^2 t - 2nx)\Big(= \vartheta_3(ix/\pi, it/\pi)\Big)$$

for $x \in \mathbb{C}$ and $\mathrm{Re}(t) > 0$. We notice that the function $\vartheta(x,t)$ satisfies the following formulas.

$$\vartheta(x + i\pi, t) = \vartheta(x,t), \tag{6.3}$$

$$\vartheta(x + t, t) = \exp(t + 2x)\vartheta(x,t), \tag{6.4}$$

$$\vartheta(x,t) = \sqrt{\frac{\pi}{t}}\exp(x^2/t)\vartheta(-i\pi x/t, \pi^2/t). \tag{6.5}$$

For simplicity we assume that $x \in \mathbb{R}$ and $t > 0$. The attached zeta function $L(s;x,t)$ of the regularized product (6.2) (if it exists) is given by

$$\begin{aligned} L(s;x,t) &= \sum_{k\in\mathbb{Z}} \vartheta(x + kt, t)^{-s} = \sum_{k\in\mathbb{Z}} \big\{\exp(k^2 t + 2kx)\vartheta(x,t)\big\}^{-s} \\ &= \vartheta(x,t)^{-s} \sum_{k\in\mathbb{Z}} \exp(-k^2 st - 2ksx) \\ &= \vartheta(x,t)^{-s}\vartheta(sx, st) \end{aligned}$$

for $\mathrm{Re}(s) > 0$. In order to see the behavior of $L(s;x,t)$ near the origin $s = 0$, we apply the theta inversion formula (6.5) and get

$$L(s;x,t) = \vartheta(x,t)^{-s}\sqrt{\frac{\pi}{st}}\exp(sx^2/t)\vartheta(-i\pi x/t, \pi^2/st).$$

We observe the contribution of the factor $\vartheta(-i\pi x/t, \pi^2/st)$. By definition $\vartheta(-i\pi x/t, \pi^2/st)$ is written as

$$\begin{aligned} \vartheta(-i\pi x/t, \pi^2/st) &= \sum_{k\in\mathbb{Z}} \exp(-\pi^2 k^2/st - 2\pi i x k/t) \\ &= 1 + \sum_{k\neq 0} \exp(-\pi^2 k^2/st - 2\pi i x k/t) \\ &= 1 + \varepsilon(s), \end{aligned}$$

where $\varepsilon(s)$ denotes an analytic function in $\mathrm{Re}(s) > 0$ which has exponential decay at $s = 0$; $s^{-N}\varepsilon(s) \to 0$ as $s \to 0$ in $\mathrm{Re}(s) > 0$ for any $N \geq 1$.

Combining the calculations above, we see that the behavior of the attached zeta function $L(s;x,t)$ of (6.2) is described as

$$\begin{aligned} L(s;x,t) &= \sqrt{\frac{\pi}{st}}(1 - s\log\vartheta(x,t) + O(s^2))(1 + sx^2/t + O(s^2))(1 + \varepsilon(s)) \\ &= \sqrt{\frac{\pi}{st}}(1 - s(\log\vartheta(x,t) - x^2/t) + O(s^2)) + \varepsilon(s) \quad (s \to 0,\ \mathrm{Re}(s) > 0)\,. \end{aligned}$$

Now we bring up a variant of the zeta regularized products motivated by this observation. This is considered to be a generalization of the dotted product $\boxdot$ (see also Remark 6.1).

Definition 6.1. Let $\psi(s)$ be a holomorphic function satisfying $\psi(0)=0$. The attached zeta function $\zeta_{\boldsymbol{a}}(s)$ of a sequence $\boldsymbol{a}=\{a_n\}_{n\in I}$ is called **asymptotically ψ-regularizable** if there exists a function $Z_{\boldsymbol{a}}(s)$ such that $Z_{\boldsymbol{a}}(s)-\zeta_{\boldsymbol{a}}(\psi(s))=\varepsilon(s)$ as $s\to 0$ in $\mathrm{Re}(s)>0$ and $Z_{\boldsymbol{a}}(s)$ is meromorphic at $s=0$. We call the function $Z_{\boldsymbol{a}}(s)$ the **asymptotically ψ-modified zeta function** of $\boldsymbol{a}$. Then, the **asymptotically ψ-regularized product** of $\boldsymbol{a}$ is defined by

$$\tag{6.6} \overset{\psi}{\underset{n\in I}{\boxdot}} a_n := \exp\left(-\operatorname*{Res}_{s=0}\frac{Z_{\boldsymbol{a}}(s)}{s^2}\right).$$

Notice that this is nothing but the dotted product $\boxdot_{n\in I} a_n$ when $\zeta_{\boldsymbol{a}}(s)$ is meromorphic at $s=0$ and $\psi(s)=s$. □

If we take $\psi(s)=s^2$, then the function $L(\psi(s);x,t)$ is asymptotically ψ-regularizable. Actually, the Laurent expansion of the asymptotically ψ-modified zeta function $\hat{L}(s;x,t)$ of $L(s;x,t)$ at $s=0$ is given by

$$\hat{L}(s;x,t)=\sqrt{\frac{\pi}{t}}\left(\frac{1}{s}-s(\log\vartheta(x,t)-x^2/t)+O(s^3)\right).$$

Hence we have

$$\operatorname*{Res}_{s=0}\frac{\hat{L}(s;x,t)}{s^2}=-\sqrt{\frac{\pi}{t}}(\log\vartheta(x,t)-x^2/t).$$

Therefore, if we employ the asymptotically ψ-regularized product, then the product (6.2) is given by

$$\tag{6.7} \Theta(x,t)=\overset{\psi}{\underset{k\in\mathbb{Z}}{\boxdot}}\vartheta(x+kt,t)=(\exp(-x^2/t)\vartheta(x,t))^{\sqrt{\pi/t}}.$$

In contrast to the translation formulas (6.3) and (6.4) of the theta function $\vartheta(x,t)$, the directions of periodicity and quasi-periodicity of $\Theta(x,t)$ are switched by taking this regularized product ${}^{\psi}\boxdot$ along the lattice $t\mathbb{Z}$. The function $\Theta(x,t)$ possesses the periodicity $\Theta(x+t,t)=\Theta(x,t)$ of the 'lattice direction' as is expected, while $\Theta(x,t)$ is quasi-periodic with respect to the period $i\pi$ as an entire function (see Section 2.2).

A similar analysis shows the

Theorem 6.1. *Let $\varphi(x,\tau)$ be a Jacobi form of weight k and index m (see, e.g.* [EZ] *for the definition). Then we have*

$$\tag{6.8} \Phi(x,\tau)=\overset{\psi}{\underset{l\in\mathbb{Z}}{\boxdot}}\varphi(x+l\tau,\tau)=\big(\exp(2\pi imx^2/\tau)\varphi(x,\tau)\big)^{\sqrt{i/2m\tau}}$$

for $x\in i\mathbb{R}$ and $\tau\in i\mathbb{R}_{>0}=\{it\in\mathbb{C}\mid t>0\}$. □

Proof. Let $\varphi(x,\tau)$ be a Jacobi form of weight k and index m. Notice that the function $\varphi(x,\tau)$ satisfies the translation formula

$$\varphi(x+l\tau,\tau)=\exp(-2m\tau\pi i l^2-4mx\pi i l)\varphi(x,\tau) \quad (k\in\mathbb{Z})\,. \tag{6.9}$$

It follows that the attached Dirichlet series of (6.8) is calculated as

$$\begin{aligned}\sum_{l\in\mathbb{Z}}\varphi(x+l\tau,\tau)^{-s}&=\varphi(x,\tau)^{-s}\sum_{l\in\mathbb{Z}}\exp(2ms\tau\pi i l^2+4msx\pi i l)\\&=\varphi(x,\tau)^{-s}\vartheta_3(2smx,2sm\tau)\\&=\varphi(x,\tau)^{-s}\sqrt{\frac{i}{2sm\tau}}\exp\left(\frac{-2\pi i s m x^2}{\tau}\right)\vartheta_3(x/\tau,-1/2sm\tau)\\&=\varphi(x,\tau)^{-s}\sqrt{\frac{i}{2sm\tau}}\exp\left(\frac{-2\pi i s m x^2}{\tau}\right)+\varepsilon(s).\end{aligned}\tag{6.10}$$

Therefore, the Laurent expansion of the asymptotically ψ-modified zeta function for $\psi(s)=s^2$ around $s=0$ is given by

$$\begin{aligned}&\varphi(x,\tau)^{-s^2}\sqrt{\frac{i}{2m\tau}}\exp\left(\frac{-2\pi i s^2 m x^2}{\tau}\right)s^{-1}\\&=\sqrt{\frac{i}{2m\tau}}\left\{\frac{1}{s}-\left(\log\varphi(x,\tau)+\frac{2\pi i m x^2}{\tau}\right)s\right\}+O(s^2).\end{aligned}$$

This implies that $\log\Phi(x,\tau)=\sqrt{\frac{i}{2m\tau}}\left(\log\varphi(x,\tau)+\frac{2\pi i m x^2}{\tau}\right)$. □

Remark 6.1. Since the function $1/\sqrt{s}$ is also rewritten as the series

$$\frac{1}{\sqrt{s}}=1+\sum_{n=1}^{\infty}\frac{2^{-n}}{n!}(\log s)^n,$$

the asymptotically ψ-modified zeta function $\hat{L}(s;x,t)$ of the product (6.2) is also expressed in the form

$$\hat{L}(s;x,t)=\underbrace{\left\{1-s(\log\vartheta(x,t)-x^2/t)+O(s^2)\right\}}_{\text{'meromorphic part'}}+\sum_{n=1}^{\infty}Q_n(s;x,t)(\log s)^n \tag{6.11}$$

for some meromorphic functions $Q_n(s;x,t)$ (this reminds us the ddotted product $\boxed{:}$). Though the (single-valued) meromorphic part in (6.11) is uniquely determined in the present case, it is not true in general. Actually, look at the formula $\sum_{n=1}^{\infty}\frac{1}{n!}(\log s)^n=s-1$ for instance.

6.2.2 Weierstrass $\wp$-function

This section gives rather experimental remarks on zeta regularized products of the elliptic functions. Let $\wp(z) = \wp(z;\omega_1,\omega_2)$ be the Weierstrass $\wp$-function

$$\wp(z) := \frac{1}{z^2} + \sum_{\substack{\omega\in\mathbb{Z}\omega_1+\mathbb{Z}\omega_2\\ \omega\neq 0}} \left\{\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right\}.$$

We want to treat a product $\prod\!\!\!\!\coprod_{n\in I}\wp(a_n x)$ for a given sequence $\{a_n\}_{n\in I}$ which converges 0, the pole of $\wp(x)$. Here we suppose that x lies in a certain compact set. Notice that $\{a_n x\}_{n\in I}$ also converges to 0. Define two Dirichlet series

$$\zeta^{\mathrm{ell}}_{\boldsymbol{a}}(s,x) := \sum_{n\in I}\wp(a_n x)^{-s},$$
$$\xi^{\mathrm{ell}}_{\boldsymbol{a}}(s,x) := \sum_{n\in I}a_n^2\wp(a_n x)^{-s}.$$

Recall the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$
$$\wp''(z) = 6\wp(z)^2 - \frac{1}{2}g_2,$$

when $g_2 = 60\sum_{\omega\neq 0}\omega^{-4}$, $g_3 = 140\sum_{\omega\neq 0}\omega^{-6}$. This yields the following assertion.

Lemma 6.2. *The Dirichlet series $\zeta^{e}_{\boldsymbol{a}}(s,x)$ and $\xi^{e}_{\boldsymbol{a}}(s,x)$ satisfy the relation*

$$\begin{aligned}\partial_x^2\zeta^{\mathrm{ell}}_{\boldsymbol{a}}(s,x) =& 2s(2s-1)\xi^{\mathrm{ell}}_{\boldsymbol{a}}(s-1,x)\\ &+\frac{g_2 s(2s+3)}{2}\xi^{\mathrm{ell}}_{\boldsymbol{a}}(s+1,x) - g_3 s(s+1)\xi^{\mathrm{ell}}_{\boldsymbol{a}}(s+2,x).\end{aligned} \tag{6.12}$$

□

Since

$$a^2\wp(ax)^{-s+1} - x^{-2}\wp(ax)^{-s} = a^2\wp(ax)^{-s}\sum_{\omega\neq 0}\left\{\frac{1}{(ax-\omega)^2} - \frac{1}{\omega^2}\right\}$$

by the definition of $\wp(z)$, we have the following.

Lemma 6.3. *We have*

$$\xi^{\mathrm{ell}}_{\boldsymbol{a}}(s-1,x) - x^{-2}\zeta^{\mathrm{ell}}_{\boldsymbol{a}}(s,x) = \sum_{n\in I}a_n^2\wp(a_n x)^{-s}\sum_{\omega\neq 0}\left\{\frac{1}{(a_n x-\omega)^2} - \frac{1}{\omega^2}\right\}.$$

In particular, for an appropriate choice of the sequence $\boldsymbol{a}$, the function $\xi^{\mathrm{ell}}_{\boldsymbol{a}}(s-1,x) - x^{-2}\zeta^{\mathrm{ell}}_{\boldsymbol{a}}(s,x)$ is holomorphic at $s=0$. □

Corollary 6.4. *The function* $(x^2\partial_x^2 - 2s(s-1))\zeta_{\boldsymbol{a}}^{\mathrm{ell}}(s,x)$ *is holomorphic and has a zero at* $s = 0$. □

Analogous to the differential equation (4.11) of the function $D_{\boldsymbol{a}}^{\mathrm{trig}}(x)$, we see that the function $D_{\boldsymbol{a}}^{\mathrm{ell}}(x) := \mathop{\prod\!\!\!\!\!\!\!\!\;:\;}_n \wp(a_n x)$ has a similar relation as follows:

Proposition 6.5. *If* $D_{\boldsymbol{a}}^{\mathrm{ell}}(x) := \mathop{\prod\!\!\!\!\!\!\!\!\;:\;}_n \wp(a_n x)$ *exists, then the function* $D_{\boldsymbol{a}}^{\mathrm{ell}}(x)$ *satisfies*

$$-x^2\partial_x^2 \log D_{\boldsymbol{a}}^{\mathrm{ell}}(x) = 4p_1(x) + 2p_0(x) + \kappa(x) \tag{6.13}$$

where $\kappa(x)$ *is given by*

$$\kappa(x) := x^2 \sum_{n\in I} a_n^2 \left\{ -2\sum_{\omega\neq 0}\left(\frac{1}{(a_n x-\omega)^2} - \frac{1}{\omega^2}\right) + \frac{3g_2}{2}\wp(a_n x)^{-1} - g_3\wp(a_n x)^{-2} \right\},$$

and the functions $\{p_k(x)\}_{k\geq -\mu}$ *are determined by the relations*

$$\begin{aligned} &x^2 p_k''(x) - 4p_{k-2}(x) + 2p_{k-1}(x) = 0 \quad (k \leq 0)\,, \\ &p_k \equiv 0 \quad (k < -\mu)\,. \end{aligned}$$

Here μ *denotes the depth of* $\zeta_{\boldsymbol{a}}^{\mathrm{ell}}(s,x)$. □

6.3 Zeta extensions

Let $f(x)$ be a zeta-like function. A function $F(x)$ is said to be a **zeta extension** of $f(x)$ if $F(x)$ satisfies a translation formula $F(x+1) = f(x)^{-1}F(x)$ [KuW1]. For instance, the higher Riemann zeta function $\zeta_{1\infty}(s)$ is a zeta extension of $\zeta(s)$; $\zeta_{1\infty}(s+1) = \zeta(s)^{-1}\zeta_{1\infty}(s)$. The zeta regularization method is effective in constructing a zeta extension from a given zeta function. Let us show several examples. Calculations are based on the property (2.2) in Proposition 2.1. We may consider the gamma function is a sort of a zeta function in the adelic sense. Thus we recall first the multiple gamma functions [B] (see also [KuKo]).

Example 6.1. A typical example is the multiple gamma functions $\Gamma_m(x)$ which are forming an ascending series of zeta extensions. In fact, the function $\Gamma_{m+1}(x)$ is a zeta extension of $\Gamma_m(x)$:

$$\begin{aligned} \Gamma_{m+1}(x) &= \prod_{k_1,\dots,k_{m+1}\geq 0} (k_1 + \cdots + k_{m+1} + x) \\ &= \prod_{k_1,\dots,k_m\geq 0} (k_1 + \cdots + k_m + x) \times \prod_{\substack{k_1,\dots,k_m\geq 0 \\ k_{m+1}\geq 1}} (k_1 + \cdots + k_{m+1} + x) \\ &= \Gamma_m(x)\Gamma_{m+1}(x+1). \end{aligned}$$

Namely we have $\Gamma_{m+1}(x+1) = \Gamma_m(x)^{-1}\Gamma_{m+1}(x)$. We note that the multiple sine functions $\mathbb{S}_m(x) := \Gamma_m(x)\Gamma_m(m-x)^{(-1)^m}$ have the same structure, that is, $\mathbb{S}_{m+1}(x)$ is a zeta extension of $\mathbb{S}_m(x)$. Note that $\mathbb{S}_3(x)^2\,\mathbb{S}_2(x)^{-3}\,\mathbb{S}_1(x)$ appears in the gamma factor of the higher Selberg zeta function $z_\Gamma(s)$ [KuW3]. □

Example 6.2. We define the function $\zeta_\infty^+(s)$, which we call a higher half Riemann zeta function, by

$$\zeta_\infty^+(s) := \prod_{\substack{\mathrm{Im}(\rho)>0 \\ m\geq 0}} (\rho + im - s).$$

This product exists and the function $\zeta_\infty^+(s)$ satisfies the functional equation

$$\zeta_\infty^+(s) = \zeta^+(s)\zeta_\infty^+(s-i),$$

where $\zeta^+(s)$ denotes the half Riemann zeta function introduced in [HKuW]. The existence of the function $\zeta_\infty^+(s)$ is shown as follows. The attached zeta function is essentially given by

$$H(w,z) := \sum_{\substack{\mathrm{Re}(\tau)>0 \\ m\geq 0}} (\tau + m - z)^{-w}.$$

This function $H(w,z)$ is also expressed as a Mellin transform

$$H(w,z) = \frac{1}{\Gamma(w)}\int_0^\infty \Phi(t)\frac{e^{zt}t^{w-1}}{1-e^{-t}}dt.$$

Here we put $\Phi(t) = \sum_{\mathrm{Re}\,\tau>0} e^{-t\tau}$ (see Section 5.2). Therefore it follows that $H(w,z)$ is a meromorphic function on $\mathbb{C}$ with respect to w since the log-singularity in $\Phi(t)$ is changed into a pole by the Mellin transform. □

Example 6.3. Retain the notation in Section 5.2. We define the function $\sin_\zeta^\alpha(x)$ by

$$\sin_\zeta^\alpha(x) := :\prod_{\substack{\mathrm{Im}(\rho)>0 \\ m\geq 0}} \sin\alpha(\rho + im - x):.$$

Here the initial domain of this normal product is taken as $\{z \in \mathbb{C} \mid 0 \leq \mathrm{Im}\, x < 2\pi/\alpha\}$. This product exists and the function $\sin_\zeta^\alpha(x)$ satisfies the functional equation

$$\sin_\zeta^\alpha(x) = S_\alpha(x)\sin_\zeta^\alpha(x-i).$$

The function $\sin_\zeta^\alpha(x)$ also possesses a quasi-periodicity on the $\mathbb{R}$-direction (see Remark 4.2). The existence of the function $\sin_\infty^\alpha(s)$ is shown as follows. The attached zeta function is essentially given by

$$\boldsymbol{L}_\alpha(s,x) := \sum_{\substack{\mathrm{Re}(\rho)>0\\ m\geq 0}} \sin\alpha(\rho+im-x)^{-s} = \sum_{m\geq 0} L_\alpha(s,x-im)$$
$$= \Phi(s\alpha)e^{i\alpha s/2}\frac{e^{f_\alpha(x)s}}{1-e^{-\alpha s}} + s\sum_{n=1}^{\infty}\frac{V(2i\alpha n)}{n}\frac{e^{-2ni\alpha x}}{1-e^{-2n\alpha}} + O(s^2).$$

This expression together with the Cramér's result for $\Phi(t)$ shows that $\boldsymbol{L}_\alpha(s,x)$ is indeed regularizable. □

We have the counterparts of Examples 6.2, 6.3 for the eigenvalues of the Laplacian Δ_Γ.

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Kazufumi KIMOTO.
Graduate School of Mathematics, Kyushu University,
Hakozaki Fukuoka 812-8581, JAPAN.
e-mail : `kimoto@math.kyushu-u.ac.jp`

Masato WAKAYAMA.
Faculty of Mathematics, Kyushu University,
Hakozaki Fukuoka 812-8581, JAPAN.
e-mail : `wakayama@math.kyushu-u.ac.jp`